Measures on tribes of fuzzy sets

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# Basic notions

## classical probability theory

<table>
<thead>
<tr>
<th>Property</th>
<th>Definition</th>
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</thead>
<tbody>
<tr>
<td>$\sigma$-algebra</td>
<td>$\mathcal{T} \subseteq 2^X$</td>
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<tr>
<td></td>
<td>$\emptyset \in \mathcal{T}$</td>
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<tr>
<td></td>
<td>$A \in \mathcal{T} \Rightarrow A' := X \setminus A \in \mathcal{T}$</td>
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<td>$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$</td>
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<td>$(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}, A_n \not\uparrow A \Rightarrow A \in \mathcal{T}$</td>
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<tr>
<td>$\sigma$-additive measure</td>
<td>$m: \mathcal{T} \rightarrow [0, \infty)$</td>
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<tr>
<td></td>
<td>$m(\emptyset) = 0$</td>
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<td></td>
<td>$m(A \cup B) = m(A) + m(B) - m(A \cap B)$</td>
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<td></td>
<td>$A_n \not\uparrow A \Rightarrow m(A_n) \rightarrow m(A)$</td>
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* $(A \cap B)(x) := A(x) \land B(x)$

** $\bullet$ is dual to $\cap$,

### fuzzy probability theory

<table>
<thead>
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<tbody>
<tr>
<td>tribe</td>
<td>$(\mathcal{T}, \land)$, where $\mathcal{T} \subseteq [0, 1]^X$</td>
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<td></td>
<td>$\land: [0, 1]^2 \rightarrow [0, 1]$ is a $t$-norm, i.e., commutative, associative, nondecreasing, and $a \land 1 = a$</td>
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<tr>
<td></td>
<td>$0 \in \mathcal{T}$</td>
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<td></td>
<td>$A \in \mathcal{T} \Rightarrow A' := 1 - A \in \mathcal{T}$</td>
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<tr>
<td></td>
<td>$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$ *</td>
</tr>
<tr>
<td></td>
<td>$(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}, A_n \not\uparrow A \Rightarrow A \in \mathcal{T}$</td>
</tr>
<tr>
<td>measure (T-measure)</td>
<td>$m: \mathcal{T} \rightarrow [0, \infty)$</td>
</tr>
<tr>
<td></td>
<td>$m(0) = 0$</td>
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<td></td>
<td>$m(A \dot{\cup} B) = m(A) + m(B) - m(A \cap B)$ **</td>
</tr>
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<td></td>
<td>$A_n \not\uparrow A \Rightarrow m(A_n) \rightarrow m(A)$</td>
</tr>
</tbody>
</table>

* $(A \cap B)'(x) := (A(x) \land B(x))'$
Underlying $\sigma$-algebra

(center, boolean part, sharp elements)

**Proposition** [Butnariu & Klement]:

Crisp elements of a tribe $(\mathcal{T}, \wedge)$, i.e., elements of $\mathcal{T} \cap \{0, 1\}^X$, determine always a $\sigma$-algebra $\mathcal{B}(\mathcal{T})$ of subsets of $X$. 
Frank family of t-norms

Frank t-norms:

\[ a \wedge_{F_s} b := \log_s \left( 1 + \frac{(s^a - 1)(s^b - 1)}{s - 1} \right) \]

for \( s \in (0, \infty) \setminus \{1\} \)

limit cases:

\( s \to 0 \quad \Rightarrow \quad a \wedge_{F_0} b := \min(a, b) \) \quad \text{standard t-norm}
\( s \to 1 \quad \Rightarrow \quad a \wedge_{F_1} b := a \cdot b \) \quad \text{product t-norm}
\( s \to \infty \quad \Rightarrow \quad a \wedge_{F_\infty} b = a \wedge_{L} b := \max(a + b - 1, 0) \) \quad \text{Łukasiewicz t-norm}

a t-norm \( \wedge \) is strict if it is continuous and \( a < b, \ 0 < c \Rightarrow a \wedge c < b \wedge c \)

a Frank t-norm \( a \wedge_{F_s} b \) is strict iff \( s \in (0, \infty) \)
Case 1: Łukasiewicz t-norm (MV-algebraic case)

**Loomis–Sikorski Theorem for MV-algebras** [Mundici, Dvurečenskij]:
Every $\sigma$-complete MV-algebra is an epimorphic image of a tribe $(\mathcal{T}, \wedge)$.

**Theorem** [Butnariu & Klement]: All elements of a tribe $(\mathcal{T}, \wedge)$ are $\mathcal{B}(\mathcal{T})$-measurable and every measure $m$ on $(\mathcal{T}, \wedge)$ is a **linear integral measure**, i.e., it is of the form

$$m(A) = \int A \, d\mu$$

where $\mu$ is a (classical) measure on $\mathcal{B}(\mathcal{T})$.

In this case, every measure is $\sigma$-additive in the usual sense.

The structure of $(\mathcal{T}, \wedge)$ may be complicated (characterized by [Klement, DN, MN]).
Case 1: Łukasiewicz t-norm II – characterization of tribes

\( \frac{k}{n} \) generates

\[ S_n := \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\} \subseteq [0, 1] \]

in particular,

\[ S_1 := \{0, 1\} \]

**Theorem:** For every tribe \((T, \wedge_L)\) there are \(\sigma\)-filters \(\nabla_n\) in \(\mathcal{B}(T)\), \(n \in N\), such that \(\nabla_m \subseteq \nabla_n\) whenever \(n\) is a divisor of \(m\),

\[ T = \{A \in [0, 1]^X : A \text{ is } \mathcal{B}(T)\text{-measurable}, \ (\forall n \in N : A^{-1}(S_n) \in \nabla_n)\} \]
Theorem [Butnariu & Klement; MN; Barbieri & H. Weber]:
Let \((\mathcal{T}, \wedge)\) be a tribe, where \(\wedge\) is a strict Frank t-norm. Then all elements of \(\mathcal{T}\) are \(\mathcal{B}(\mathcal{T})\)-measurable and every measure \(m\) on \((\mathcal{T}, \wedge)\) is of the form

\[
m(A) = \mu(\text{Supp } A) + \int A \, d\nu
\]

where \(\text{Supp } A := \{x \in X : A(x) > 0\}\) and \(\mu, \nu\) are (classical) measures on \(\mathcal{B}(\mathcal{T})\).

\(\mu(\text{Supp } A)\) ... support measure

Example [Barbieri & H. Weber]: A measure need not be monotonic, e.g.,

\[
m(A) = \mu(\text{Supp } A) - \int A \, d\mu
\]
**Case 3: Nearly Frank t-norms**

**Nearly Frank t-norm** is a t-norm which can be written as

\[ a \wedge_N b := h^{-1}(h(a) \wedge_{F_s} h(b)) \]

where \( \wedge \) is a Frank t-norm and

\( h: [0, 1] \rightarrow [0, 1] \) is a **negation-preserving automorphism (NPA)**, i.e., an increasing bijection satisfying \( h(a') = (h(a))' \)

a NPA is uniquely determined by its restriction to \([0, 1/2]\)

the inverse of a NPA is a NPA

\( \wedge \) and \( h \) are uniquely determined by \( \wedge \)

nearly Frank \( \not\Rightarrow \) Frank (take \( h \neq id \))
Theorem [Barbieri & MN & H. Weber]:
Let \((\mathcal{T}, \wedge)\) be a tribe, where \(\wedge\) is a strict nearly Frank t-norm. Then all elements of \(\mathcal{T}\) are \(\mathcal{B}(\mathcal{T})\)-measurable and every measure \(m\) on \((\mathcal{T}, \wedge)\) is of the form

\[
m(A) = \mu(\text{Supp } A) + \int (h \circ A) \, d\nu
\]

where \(\mu, \nu\) are (classical) measures on \(\mathcal{B}(\mathcal{T})\).

\[
\int (h \circ A) \, d\nu \quad \text{... generalized integral measure}
\]
Generated and weakly generated tribes

[Butnariu & Klement; Mesiar; MN]

**Problem:** For a general strict t-norm $\wedge$, we do not know whether all elements of a tribe $(\mathcal{T}, \wedge)$ are $\mathcal{B}(\mathcal{T})$-measurable.

⇒ **Additional assumption:**
$\mathcal{T}$ is a **weakly generated tribe**, i.e., there is a $\sigma$-filter $\nabla$ in $\mathcal{B}(\mathcal{T})$ such that $\mathcal{T}$ is the collection of all functions $A: X \to [0, 1]$ satisfying

- $A$ is $\mathcal{B}(\mathcal{T})$-measurable,
- $A^{-1}(\{0, 1\}) \in \nabla$.

**Particular case:** ($\nabla = \mathcal{B}(\mathcal{T})$)
$\mathcal{T}$ is a **generated tribe** if $\mathcal{T}$ is the collection of all $\mathcal{B}(\mathcal{T})$-measurable functions $A: X \to [0, 1]$

**Open problem:** Is there a strict t-norm $\wedge$ and a tribe $(\mathcal{T}, \wedge)$ which is not weakly generated?

**Partial answer:** Not for nearly Frank and for many other t-norms.
Case 4: Strict t-norms which are not nearly Frank

**Proposition [MN]:** They exist.

**Problem [MN]:** Recognize them! (By an algorithm.)

**Solved** by [Mesiar].

Characterization of measures obtained [MN] for monotonic $T$-measures on generated tribes.

Now more generally, for nonmonotonic $T$-measures on weakly generated tribes:

**Theorem [Barbieri & MN & H. Weber]:**
Let $(\mathcal{T}, \wedge)$ be a tribe, where $\wedge$ is a strict t-norm which is not nearly Frank. Then every measure $m$ on $(\mathcal{T}, \wedge)$ is a support measure, i.e., it is of the form

$$m(A) = \mu(\text{Supp} A)$$

where $\mu$ is a (classical) measure on $\mathcal{B}(\mathcal{T})$. 