Regular measures on tribes of fuzzy sets

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Related work presented at Linz Seminars

1979

Henri M. Prade: Nomenclature of fuzzy measures

Erich Peter Klement: Extension of probability measures to fuzzy measures and their characterization

Werner Schwyhla: Conditions for a fuzzy probability measure to be an integral

Josette and Jean-Louis Coulon: Fuzzy boolean algebras

1980

Erich Peter Klement: Some remarks on t-norms, fuzzy σ-algebras and fuzzy measures

Werner Schwyhla: Remarks on non-additive measures and fuzzy sets

Ulrich Höhle: Fuzzy measures as extensions

1981

Erich Peter Klement: Fuzzy measures assuming their values in the set of fuzzy numbers

1982

Erich Peter Klement: On different approaches to fuzzy probabilities

Didier Dubois: Upper and lower possibilistic expectations and applications

Ronald R. Yager: Probabilities from fuzzy observations

1983

Siegfried Weber: How to measure fuzzy sets

1984

Robert Lowen: Spaces of probability measures revisited

1985

Siegfried Weber: Generalizing the axioms of probability

1986

Erich Peter Klement: Representation of crisp- and fuzzy-valued measures by integrals

Siegfried Weber: Some remarks on the theory of pseudo-additive measures and its applications

1987
Erich Peter Klement: On a class of non-additive measures and integrals

1988
Alain Chateauneuf: Decomposable measures, distorted probabities and concave capacities
Siegfried Weber: Decomposable measures for conditional objects
Aldo Ventre: A Yosida-Hewitt like theorem for $\perp$-decomposable measures (joint paper with M. Squillante)
Massimo Squillante: $\perp$-decomposable measures and integrals: Convergence and absolute continuity (joint paper with L. D'Apuzzo and R. Sarno)
Ulrich Höhle: Non-classical models of probability theory

$\vdots$

1998

Mirko Navara, Pavel Pták: Types of uncertainty and the role of the Frank t-norms in classical and nonclassical logics

Mirko Navara: Nearly Frank t-norms and the characterization of T-measures

Giuseppina Barbieri: A representation theorem and a Liapounoff theorem for T_s-measures

Beloslav Riečan: On the probability theory and fuzzy sets

Ulrich Höhle: Realizations for generalized probability measures

Marc Roubens: On probabilistic interactions among players in cooperative games

Radko Mesiar: k-order pseudo-additive measures

⋮

Classical measure theory [Halmos]

THEOREMS about

FUNCTIONALS (MEASURES) on

SETS

Also [Sugeno; Dubois, Prade; Wang, Klir; Pap]

What is fuzzy measure theory?

THEOREMS about

FUZZY FUNCTIONALS (MEASURES) on

SETS

[Feng; Guo, Zhang, Wu]

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[Butnariu, Klement]

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[Butnariu, Klement, Mesiar, Barbieri, Weber]

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[Butnariu, Klement, Mesiar, Barbieri, Weber]

Also measure theory on MV-algebras [Cignoli, D'Ottaviano, Mundici, Riečan]

Basic fuzzy logical operations

Standard negation, $\neg x = 1 - x$

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Fuzzy disjunction (t-conorm): $S\colon [0,1]^2 \to [0,1]$ dual to T:

$$S(x,y) = \neg T(\neg x, \neg y)$$

Basic notions of fuzzy measure theory

classical measure theory	**fuzzy** measure theory
σ-algebra $\mathcal{T} \subseteq \mathbf{2}^X$	**tribe** $(\mathcal{T}, T)$, where $\mathcal{T} \subseteq [\mathbf{0}, \mathbf{1}]^X$
$\emptyset \in \mathcal{T}$	$0 \in \mathcal{T}$
$A \in \mathcal{T} \Rightarrow A' = X \setminus A \in \mathcal{T}$	$A \in \mathcal{T} \Rightarrow A' = 1 - A \in \mathcal{T}$
$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$	$A, B \in \mathcal{T} \Rightarrow A \underset{\cdot}{\cap} B \in \mathcal{T}$ *
$(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}, A_n \nearrow A \Rightarrow A \in \mathcal{T}$	$(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}, A_n \nearrow A \Rightarrow A \in \mathcal{T}$
measure $\mu : \mathcal{T} \to [0, \infty[$	**measure** $\mu : \mathcal{T} \to [0, \infty[$
$\mu(\emptyset) = 0$	$\mu(0) = 0$
$\mu(A \cup B)$	$\mu(A \dot{\cup} B)$
$= \mu(A) + \mu(B) - \mu(A \cap B)$	$= \mu(A) + \mu(B) - \mu(A \underset{\cdot}{\cap} B)$ *
$A_n \nearrow A \Rightarrow \mu(A_n) \to \mu(A)$	$A_n \nearrow A \Rightarrow \mu(A_n) \to \mu(A)$

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Always: Crisp elements of $\mathcal{T}$, i.e., $\mathcal{T} \cap \{0, 1\}^X$, determine a σ-algebra $\mathcal{B}$

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Full tribes

Example: Let $\mathcal{B}$ be a σ-algebra of subsets of X,
$\mathcal{T}$ be the corresponding collection of characteristic functions (indicators):

$$\mathcal{T} = \{\chi_A \mid A \in \mathcal{B}\}\,.$$

Then $(\mathcal{T}, T)$ is a tribe (for any t-norm T).
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Example: Let $\mathcal{B}$ be a σ-algebra of subsets of X,
$\mathcal{T} = \{A \in [0,1]^X \mid A \text{ is } \mathcal{B}\text{-measurable}\}$
Then $(\mathcal{T}, T)$ is a T-tribe for any measurable t-norm T.
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Łukasiewicz t-norm

$$T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$$

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Theorem: [Butnariu, Klement] All elements of $\mathcal{T}$ are $\mathcal{B}$-measurable. Each measure is **regular** and it is of the form

$$\mu(A) = \int A \, d\nu$$

where $\nu = \mu \restriction \mathcal{B}$ is a (classical) measure on $\mathcal{B}$.

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$\int A\, d\nu$ ... **linear integral measure**

Frank t-norms

Frank t-norms $T^{\mathbf{F}}_{\lambda}$, $\lambda \in [0,\infty]$, [Frank] are defined by

$$T^{\mathbf{F}}_{\lambda}(x,y) = \begin{cases} \log_{\lambda}\left(1 + \frac{(\lambda^{x}-1)(\lambda^{y}-1)}{\lambda-1}\right) & \text{if } \lambda \in \,]0,\infty[\, \setminus \{1\}\,, \\ T_{\mathbf{M}}(x,y) = \min(x,y) & \text{if } \lambda = 0\,, \\ T_{\mathbf{P}}(x,y) = x \cdot y & \text{if } \lambda = 1\,, \\ T_{\mathbf{L}}(x,y) = \max(x+y-1,0) & \text{if } \lambda = \infty\,. \end{cases}$$

Regular measures on full tribes, strict Frank t-norms

Frank t-norm $T^{\mathbf{F}}_{\lambda}$ is strict iff $0 < \lambda < \infty$

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Theorem: Regular measures on $(\mathcal{T}, T^{\mathbf{F}}_{\lambda})$ are (regular) measures on $(\mathcal{T}, T_{\mathbf{L}})$, i.e., of the form

$$\mu(A) = \int A \, d\nu$$

where $\nu = \mu \restriction \mathcal{B}$ is a classical measure on $\mathcal{B}$ (μ is a linear integral measure).

Nearly Frank t-norms

[Mesiar, MN]

Nearly Frank t-norm:

$$T(a,b) = h_T^{-1}(T^{\mathbf{F}}_{\lambda_T}(h_T(a), h_T(b)))$$

where $T^{\mathbf{F}}_{\lambda_T}$ is a Frank t-norm and
$h_T\colon [0,1] \to [0,1]$ is an increasing bijection which **commutes with** $\neg$, i.e.,
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There are nearly Frank t-norms which are not Frank (take $h_T \neq id$)

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$\int (h_T \circ A)\, d\nu$ ... **generalized integral measure**

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measures	tribes	T nearly Frank (with h_T)	T not nearly Frank
regular	full	$\int (h_T \circ A)\, d\nu$	0

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$\varrho(\mathrm{Supp}\, A)$ ... **support measure**

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Theorem: [Butnariu, Klement; Mesiar, MN] Each measure is of the form

$$\mu(A) = \int (h_T \circ A)\, d\nu \pm \varrho(\mathrm{Supp}\, A)$$

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Example:

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It is a support measure on a singleton.

It is **monotone**, but **not regular**.

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Example:

$$\mu(x) = \begin{cases} 1 - \frac{x}{2} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It is a measure which is a linear combination of a support measure κ and a regular measure $\nu = id$, $\mu = \kappa - \frac{1}{2} id$

It is **neither monotone nor regular**.

Charges

Alternative notion:

Signed measure (**charge**) on a tribe $(\mathcal{T}, T)$:
$\mu\colon \mathcal{T} \to \mathbb{R}$ s.t.

- $\mu(0) = 0$
- $\mu(A \mathbin{\dot\cup} B) = \mu(A) + \mu(B) - \mu(A \mathbin{\underset{\cdot}{\cap}} B)$
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Moreover, for a **regular signed measure** (**regular charge**) we require

- $A_n \searrow A \Rightarrow \mu(A_n) \to \mu(A)$

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$\mathcal{T} = \{A \in [0,1]^{\mathbb{R}} \mid A$ is Borel-measurable, $A(x) \in \{0,1\}$ almost everywhere$\}$
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Example: [Butnariu, Klement; Mesiar; MN]
$\mathcal{B}$ ... a σ-algebra of subsets of X
Δ ... a σ-ideal in $\mathcal{B}$
$\mathcal{T} = \{A \in [0,1]^X \mid A$ is $\mathcal{B}$-measurable, $A^{-1}(]0,1[) \in \Delta\}$
$(\mathcal{T}, T)$ is a tribe called a **weakly full tribe** (**weakly generated tribe**)

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Example: Strict t-norms from the following families are sufficient: Aczél–Alsina, Frank, Hamacher*, the eighth Mizumoto, the tenth Mizumoto, Schweizer–Sklar*, the third Schweizer, etc.

* except for one value of the parameter

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$$T_0^{\mathbf{H}}(x,y) = \begin{cases} 0 & \text{if } x = y = 0\,, \\ \frac{x\,y}{x+y-x\,y} & \text{otherwise}\,. \end{cases}$$

Let $X = \{x, y\}$. There is a distance d on $[0,1]$ and $c \in \mathbb{R}$ such that $(\mathcal{T}, T_0^{\mathbf{H}})$ is a tribe, where $\mathcal{T} = \{0,1\} \cup \{A \in \left]0,1\right[^X \mid d(A(x), A(y)) \le c\}$

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Further counterexamples:
T-norms T obtained from the Hamacher product by the formula

$$T(x,y) = h_T^{-1}\left(T_0^{\mathbf{H}}(h_T(x), h_T(y))\right),$$

where h_T is an order automorphism of $[0,1]$ which commutes with $\neg$

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- strict Dombi t-norms $(\lambda \in]0, \infty[)$

$$x \underset{D_\lambda}{\wedge} y = \frac{1}{\left(\left(\frac{1}{x} - 1\right)^{\lambda} + \left(\frac{1}{y} - 1\right)^{\lambda}\right)^{\frac{1}{\lambda}} + 1}$$

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- the first Mizumoto t-norm

$$x \underset{M1}{\wedge} y = \frac{2}{\pi} \operatorname{arccot}\left(\cot \frac{\pi}{2} x + \cot \frac{\pi}{2} y\right)$$

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- strict Dombi t-norms $(\lambda \in]0, \infty[)$

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- the first Mizumoto t-norm

$$x \underset{M1}{\wedge} y = \frac{2}{\pi} \operatorname{arccot}\left(\cot \frac{\pi}{2} x + \cot \frac{\pi}{2} y\right)$$

Theorem: All other strict t-norms **found in the literature** are sufficient.

Tribes which are not weakly full

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General case: something between.

Regular measures on tribes

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all	full	$\int (h_T \circ A)\, d\nu \pm \varrho(\operatorname{Supp} A)$	$\varrho(\operatorname{Supp} A)$
regular	general	$\int (h_T \circ A)\, d\nu$	$\varrho(\operatorname{Supp} A)$ *

* where $\varrho \restriction \Delta_{\mathcal{T}} = 0$; then $\varrho(\operatorname{Supp} A) = \varrho(A^{-1}(1)) = \int (h_T \circ A)\, d\varrho$ (h_T arbitrary)

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For **regular** measures, the characterization is known in full generality.

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Frank (more exactly, nearly Frank) t-norms play a prominent role in the characterization of measures.