

Regular measures on tribes of fuzzy sets

Mirko Navara and Pavel Pták

Center for Machine Perception and Department of Mathematics

Faculty of Electrical Engineering

Czech Technical University

166 27 Praha, Czech Republic

navara@cmp.felk.cvut.cz, ptak@math.feld.cvut.cz

<http://cmp.felk.cvut.cz/~navara>



Related work presented at Linz Seminars

1979

Henri M. Prade: Nomenclature of fuzzy measures

Erich Peter Klement: Extension of probability measures to fuzzy measures and their characterization

Werner Schwyhla: Conditions for a fuzzy probability measure to be an integral

Josette and Jean-Louis Coulon: Fuzzy boolean algebras

1980

Erich Peter Klement: Some remarks on t-norms, fuzzy σ -algebras and fuzzy measures

Werner Schwyhla: Remarks on non-additive measures and fuzzy sets

Ulrich Höhle: Fuzzy measures as extensions

1981

Erich Peter Klement: Fuzzy measures assuming their values in the set of fuzzy numbers

1982

Erich Peter Klement: On different approaches to fuzzy probabilities

Didier Dubois: Upper and lower possibilistic expectations and applications

Ronald R. Yager: Probabilities from fuzzy observations

1983

Siegfried Weber: How to measure fuzzy sets

1984

Robert Lowen: Spaces of probability measures revisited

1985

Siegfried Weber: Generalizing the axioms of probability

1986

Erich Peter Klement: Representation of crisp- and fuzzy-valued measures by integrals

Siegfried Weber: Some remarks on the theory of pseudo-additive measures and its applications

1987

Erich Peter Klement: On a class of non-additive measures and integrals

1988

Alain Chateauneuf: Decomposable measures, distorted probabilities and concave capacities

Siegfried Weber: Decomposable measures for conditional objects

Aldo Venturelli: A Yosida-Hewitt like theorem for \perp -decomposable measures (joint paper with M. Squillante)

Massimo Squillante: \perp -decomposable measures and integrals: Convergence and absolute continuity (joint paper with L. D'Apuzzo and R. Sarno)

Ulrich Höhle: Non-classical models of probability theory

⋮

1998

Mirko Navara, Pavel Pták: Types of uncertainty and the role of the Frank t-norms in classical and nonclassical logics

Mirko Navara: Nearly Frank t-norms and the characterization of T -measures

Giuseppina Barbieri: A representation theorem and a Liapounoff theorem for T_s -measures

Beloslav Riečan: On the probability theory and fuzzy sets

Ulrich Höhle: Realizations for generalized probability measures

Marc Roubens: On probabilistic interactions among players in cooperative games

Radko Mesiar: k -order pseudo-additive measures

⋮

Classical measure theory [Halmos]

THEOREMS about

FUNCTIONALS (MEASURES) on

SETS

Also [Sugeno; Dubois, Prade; Wang, Klir; Pap]

What is fuzzy measure theory?

THEOREMS about

FUZZY FUNCTIONALS (MEASURES) on

SETS

[Feng; Guo, Zhang, Wu]

What is fuzzy measure theory?

THEOREMS about

FUNCTIONALS (MEASURES) on

FUZZY SETS

[Butnariu, Klement]

What is fuzzy measure theory?

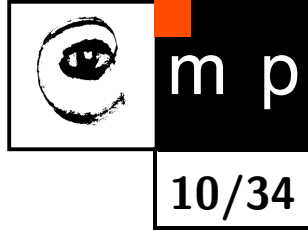
THEOREMS about

FUZZY FUNCTIONALS (MEASURES) on

FUZZY SETS

[???

What is fuzzy measure theory?



FUZZY THEOREMS about

FUZZY FUNCTIONALS (MEASURES) on

FUZZY SETS

[!!!]

What is fuzzy measure theory?

HERE:

THEOREMS about

FUNCTIONALS (MEASURES) on

FUZZY SETS

[Butnariu, Klement, Mesiar, Barbieri, Weber]

What is fuzzy measure theory?

HERE:

THEOREMS about

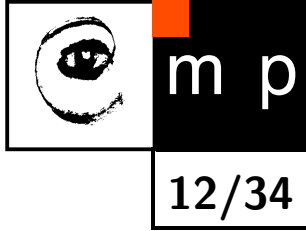
FUNCTIONALS (MEASURES) on

FUZZY SETS

[Butnariu, Klement, Mesiar, Barbieri, Weber]

Also measure theory on MV-algebras [Cignoli, D'Ottaviano, Mundici, Riečan]

Basic fuzzy logical operations



Standard negation, $\neg x = 1 - x$

Basic fuzzy logical operations

Standard negation, $\neg x = 1 - x$

Fuzzy conjunction (t-norm): $T: [0, 1]^2 \rightarrow [0, 1]$ which is commutative, associative, nondecreasing, and $T(a, 1) = a$

Basic fuzzy logical operations

Standard negation, $\neg x = 1 - x$

Fuzzy conjunction (t-norm): $T: [0, 1]^2 \rightarrow [0, 1]$ which is commutative, associative, nondecreasing, and $T(a, 1) = a$

A t-norm T is **strict** iff it is **continuous** and
 $x > y, z > 0 \Rightarrow T(x, z) > T(y, z)$

Basic fuzzy logical operations

Standard negation, $\neg x = 1 - x$

Fuzzy conjunction (t-norm): $T: [0, 1]^2 \rightarrow [0, 1]$ which is commutative, associative, nondecreasing, and $T(a, 1) = a$

A t-norm T is **strict** iff it is **continuous** and $x > y, z > 0 \Rightarrow T(x, z) > T(y, z)$

Fuzzy disjunction (t-conorm): $S: [0, 1]^2 \rightarrow [0, 1]$ dual to T :

$$S(x, y) = \neg T(\neg x, \neg y)$$

Basic notions of fuzzy measure theory

classical measure theory	fuzzy measure theory
<p>σ-algebra $\mathcal{T} \subseteq 2^X$</p> <p>$\emptyset \in \mathcal{T}$</p> <p>$A \in \mathcal{T} \Rightarrow A' = X \setminus A \in \mathcal{T}$</p> <p>$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$</p> <p>$(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}, A_n \nearrow A \Rightarrow A \in \mathcal{T}$</p>	<p>tribe (\mathcal{T}, T), where $\mathcal{T} \subseteq [0, 1]^X$</p> <p>$0 \in \mathcal{T}$</p> <p>$A \in \mathcal{T} \Rightarrow A' = 1 - A \in \mathcal{T}$</p> <p>$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$ *</p> <p>$(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}, \overset{\cdot}{A}_n \nearrow A \Rightarrow A \in \mathcal{T}$</p>
<p>measure $\mu: \mathcal{T} \rightarrow [0, \infty[$</p> <p>$\mu(\emptyset) = 0$</p> <p>$\mu(A \cup B)$ $= \mu(A) + \mu(B) - \mu(A \cap B)$</p> <p>$A_n \nearrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$</p>	<p>measure $\mu: \mathcal{T} \rightarrow [0, \infty[$</p> <p>$\mu(0) = 0$</p> <p>$\mu(A \overset{\cdot}{\cup} B)$ $= \mu(A) + \mu(B) - \mu(A \overset{\cdot}{\cap} B)$ *</p> <p>$A_n \nearrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$</p>

* $(A \overset{\cdot}{\cap} B)(x) = T(A(x), B(x)), \quad (A \overset{\cdot}{\cup} B)(x) = S(A(x), B(x))$

Basic notions of fuzzy measure theory

classical measure theory	fuzzy measure theory
<p>σ-algebra $\mathcal{T} \subseteq 2^X$</p> <p>$\emptyset \in \mathcal{T}$</p> <p>$A \in \mathcal{T} \Rightarrow A' = X \setminus A \in \mathcal{T}$</p> <p>$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$</p> <p>$(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}, A_n \nearrow A \Rightarrow A \in \mathcal{T}$</p>	<p>tribe (\mathcal{T}, T), where $\mathcal{T} \subseteq [0, 1]^X$</p> <p>$0 \in \mathcal{T}$</p> <p>$A \in \mathcal{T} \Rightarrow A' = 1 - A \in \mathcal{T}$</p> <p>$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T} *$</p> <p>$(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}, A_n \nearrow A \Rightarrow A \in \mathcal{T}$</p>
<p>measure $\mu: \mathcal{T} \rightarrow [0, \infty[$</p> <p>$\mu(\emptyset) = 0$</p> <p>$\mu(A \cup B)$ $= \mu(A) + \mu(B) - \mu(A \cap B)$</p> <p>$A_n \nearrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$</p> <p>$A_n \searrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$</p>	<p>regular measure $\mu: \mathcal{T} \rightarrow [0, \infty[$</p> <p>$\mu(0) = 0$</p> <p>$\mu(A \dot{\cup} B)$ $= \mu(A) + \mu(B) - \mu(A \cap B) *$</p> <p>$A_n \nearrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$</p> <p>$A_n \searrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$</p>

* $(A \cap B)(x) = T(A(x), B(x)), \quad (A \dot{\cup} B)(x) = S(A(x), B(x))$

Basic notions of fuzzy measure theory

classical measure theory	fuzzy measure theory
<p>σ-algebra $\mathcal{T} \subseteq 2^X$</p> <p>$\emptyset \in \mathcal{T}$</p> <p>$A \in \mathcal{T} \Rightarrow A' = X \setminus A \in \mathcal{T}$</p> <p>$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$</p> <p>$(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}, A_n \nearrow A \Rightarrow A \in \mathcal{T}$</p>	<p>tribe (\mathcal{T}, T), where $\mathcal{T} \subseteq [0, 1]^X$</p> <p>$0 \in \mathcal{T}$</p> <p>$A \in \mathcal{T} \Rightarrow A' = 1 - A \in \mathcal{T}$</p> <p>$A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$ *</p> <p>$(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}, \overset{\cdot}{A}_n \nearrow A \Rightarrow A \in \mathcal{T}$</p>
<p>measure $\mu: \mathcal{T} \rightarrow [0, \infty[$</p> <p>$\mu(\emptyset) = 0$</p> <p>$\mu(A \cup B)$ $= \mu(A) + \mu(B) - \mu(A \cap B)$</p> <p>$A_n \nearrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$</p> <p>$A_n \searrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$</p>	<p>regular measure $\mu: \mathcal{T} \rightarrow [0, \infty[$</p> <p>$\mu(0) = 0$</p> <p>$\mu(A \overset{\cdot}{\cup} B)$ $= \mu(A) + \mu(B) - \mu(A \cap B)$ *</p> <p>$A_n \nearrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$</p> <p>$A_n \searrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$</p>

Always: Crisp elements of \mathcal{T} , i.e., $\mathcal{T} \cap \{0, 1\}^X$, determine a σ -algebra \mathcal{B}

* $(A \overset{\cdot}{\cap} B)(x) = T(A(x), B(x)), \quad (A \overset{\cdot}{\cup} B)(x) = S(A(x), B(x))$

Full tribes

Example: Let \mathcal{B} be a σ -algebra of subsets of X ,
 \mathcal{T} be the corresponding collection of characteristic functions (indicators):

$$\mathcal{T} = \{\chi_A \mid A \in \mathcal{B}\}.$$

Then (\mathcal{T}, T) is a tribe (for any t-norm T).
It is called a **Boolean tribe**.

Full tribes

Example: Let \mathcal{B} be a σ -algebra of subsets of X ,
 \mathcal{T} be the corresponding collection of characteristic functions (indicators):

$$\mathcal{T} = \{\chi_A \mid A \in \mathcal{B}\}.$$

Then (\mathcal{T}, T) is a tribe (for any t-norm T).

It is called a **Boolean tribe**.

Example: The tribe of all constants from $[0, 1]$
 (w.l.o.g., with a singleton domain)
 may be identified with numbers from $[0, 1]$.

It is called a **full tribe of constants**.

Full tribes

Example: Let \mathcal{B} be a σ -algebra of subsets of X ,
 \mathcal{T} be the corresponding collection of characteristic functions (indicators):

$$\mathcal{T} = \{\chi_A \mid A \in \mathcal{B}\}.$$

Then (\mathcal{T}, T) is a tribe (for any t-norm T).
 It is called a **Boolean tribe**.

Example: The tribe of all constants from $[0, 1]$
 (w.l.o.g., with a singleton domain)
 may be identified with numbers from $[0, 1]$.
 It is called a **full tribe of constants**.

Example: Let \mathcal{B} be a σ -algebra of subsets of X ,
 $\mathcal{T} = \{A \in [0, 1]^X \mid A \text{ is } \mathcal{B}\text{-measurable}\}$
 Then (\mathcal{T}, T) is a T -tribe for any measurable t-norm T .
 It is called a **full tribe**.

Łukasiewicz t-norm

$$T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$$

These tribes correspond to set-representable σ -complete MV-algebras

Łukasiewicz t-norm

$$T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$$

These tribes correspond to set-representable σ -complete MV-algebras

Theorem: [Butnariu, Klement] All elements of \mathcal{T} are \mathcal{B} -measurable. Each measure is **regular** and it is of the form

$$\mu(A) = \int A d\nu$$

where $\nu = \mu \upharpoonright \mathcal{B}$ is a (classical) measure on \mathcal{B} .

Łukasiewicz t-norm

$$T_L(x, y) = \max(x + y - 1, 0)$$

These tribes correspond to set-representable σ -complete MV-algebras

Theorem: [Butnariu, Klement] All elements of \mathcal{T} are \mathcal{B} -measurable. Each measure is **regular** and it is of the form

$$\mu(A) = \int A d\nu$$

where $\nu = \mu \upharpoonright \mathcal{B}$ is a (classical) measure on \mathcal{B} .

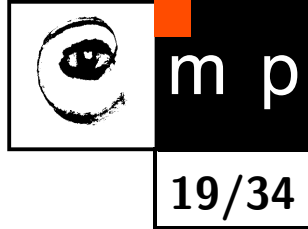
$\int A d\nu$... **linear integral measure**

Frank t-norms

Frank t-norms $T_\lambda^{\mathbf{F}}$, $\lambda \in [0, \infty]$, [Frank] are defined by

$$T_\lambda^{\mathbf{F}}(x, y) = \begin{cases} \log_\lambda \left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1} \right) & \text{if } \lambda \in]0, \infty[\setminus \{1\}, \\ T_{\mathbf{M}}(x, y) = \min(x, y) & \text{if } \lambda = 0, \\ T_{\mathbf{P}}(x, y) = x \cdot y & \text{if } \lambda = 1, \\ T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0) & \text{if } \lambda = \infty. \end{cases}$$

Regular measures on full tribes, strict Frank t-norms



Frank t-norm T_λ^F is strict iff $0 < \lambda < \infty$

Regular measures on full tribes, strict Frank t-norms

Frank t-norm T_λ^F is strict iff $0 < \lambda < \infty$

Theorem: Regular measures on $(\mathcal{T}, T_\lambda^F)$ are (regular) measures on (\mathcal{T}, T_L) ,
i.e., of the form

$$\mu(A) = \int A d\nu$$

where $\nu = \mu \upharpoonright \mathcal{B}$ is a classical measure on \mathcal{B}
(μ is a linear integral measure).

Nearly Frank t-norms

[Mesiar, MN]

Nearly Frank t-norm:

$$T(a, b) = h_T^{-1}(T_{\lambda_T}^{\mathbf{F}}(h_T(a), h_T(b)))$$

where $T_{\lambda_T}^{\mathbf{F}}$ is a Frank t-norm and

$h_T: [0, 1] \rightarrow [0, 1]$ is an increasing bijection which **commutes with** \neg , i.e.,

$$h_T(\neg a) = \neg h_T(a)$$

Nearly Frank t-norms

[Mesiar, MN]

Nearly Frank t-norm:

$$T(a, b) = h_T^{-1}(T_{\lambda_T}^{\mathbf{F}}(h_T(a), h_T(b)))$$

where $T_{\lambda_T}^{\mathbf{F}}$ is a Frank t-norm and

$h_T: [0, 1] \rightarrow [0, 1]$ is an increasing bijection which **commutes with** \neg , i.e.,

$$h_T(\neg a) = \neg h_T(a)$$

λ_T, h_T are uniquely determined by T (except for the case $T = T_{\mathbf{M}}$)

Nearly Frank t-norms

[Mesiar, MN]

Nearly Frank t-norm:

$$T(a, b) = h_T^{-1}(T_{\lambda_T}^{\mathbf{F}}(h_T(a), h_T(b)))$$

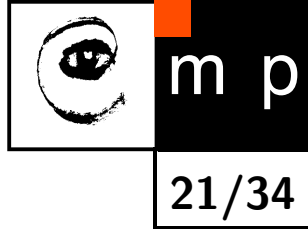
where $T_{\lambda_T}^{\mathbf{F}}$ is a Frank t-norm and

$h_T: [0, 1] \rightarrow [0, 1]$ is an increasing bijection which **commutes with** \neg , i.e.,
 $h_T(\neg a) = \neg h_T(a)$

λ_T, h_T are uniquely determined by T (except for the case $T = T_{\mathbf{M}}$)

There are nearly Frank t-norms which are not Frank (take $h_T \neq id$)

Regular measures on **full** tribes,
strict nearly Frank t-norms



Strict nearly Frank t-norms correspond to **strict** Frank t-norms

Regular measures on full tribes, strict nearly Frank t-norms

Strict nearly Frank t-norms correspond to strict Frank t-norms

Theorem: Each regular measure is of the form

$$\mu(A) = \int (h_T \circ A) d\nu$$

where $\nu = \mu \upharpoonright \mathcal{B}$ is a classical measure on \mathcal{B} .

Regular measures on full tribes, strict nearly Frank t-norms

Strict nearly Frank t-norms correspond to strict Frank t-norms

Theorem: Each regular measure is of the form

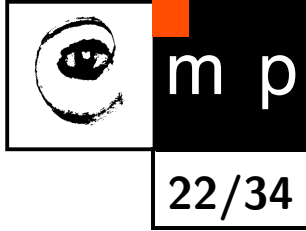
$$\mu(A) = \int (h_T \circ A) d\nu$$

where $\nu = \mu \upharpoonright \mathcal{B}$ is a classical measure on \mathcal{B} .

$\int (h_T \circ A) d\nu$... **generalized integral measure**

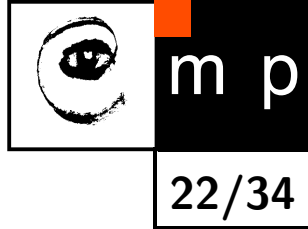
Regular measures on **full** tribes,
strict t-norms which are **not nearly Frank**

**Regular measures on full tribes,
strict t-norms which are not nearly Frank**



There are strict t-norms which are not nearly Frank [MN]

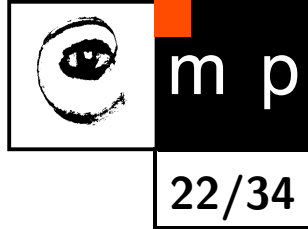
Regular measures on full tribes, strict t-norms which are not nearly Frank



There are strict t-norms which are not nearly Frank [MN]

Can we recognize them?

Regular measures on full tribes, strict t-norms which are not nearly Frank



There are strict t-norms which are not nearly Frank [MN]

Can we recognize them?

Yes [Mesiar]

Regular measures on full tribes, strict t-norms which are not nearly Frank



There are strict t-norms which are not nearly Frank [MN]

Can we recognize them?

Yes [Mesiar]

Theorem: For each strict t-norm which is not nearly Frank, there are **no non-zero regular measures** on a full tribe.

Regular measures on full tribes, strict t-norms which are not nearly Frank

There are strict t-norms which are not nearly Frank [MN]

Can we recognize them?

Yes [Mesiar]

Theorem: For each strict t-norm which is not nearly Frank, there are **no non-zero regular measures** on a full tribe.

measures	tribes	T nearly Frank (with h_T)	T not nearly Frank
regular	full	$\int (h_T \circ A) d\nu$	0

Measures on **full** tribes,

strict t-norms which are not nearly Frank

(Regularity is omitted.)

Measures on full tribes, strict t-norms which are not nearly Frank

(Regularity is omitted.)

Theorem: Each measure is of the form

$$\mu(A) = \varrho(\text{Supp } A)$$

where ϱ is a classical measure on \mathcal{B} .

Measures on full tribes, strict t-norms which are not nearly Frank

(Regularity is omitted.)

Theorem: Each measure is of the form

$$\mu(A) = \varrho(\text{Supp } A)$$

where ϱ is a classical measure on \mathcal{B} .

$\varrho(\text{Supp } A)$... **support measure**

Measures on **full** tribes,
strict **nearly Frank** t-norms

Measures on full tribes, strict nearly Frank t-norms

Theorem: [Butnariu, Klement; Mesiar, MN] Each measure is of the form

$$\mu(A) = \int (h_T \circ A) d\nu \pm \varrho(\text{Supp } A)$$

where ν, ϱ are classical measures on \mathcal{B} .

Measures on full tribes, strict nearly Frank t-norms

Theorem: [Butnariu, Klement; Mesiar, MN] Each measure is of the form

$$\mu(A) = \int (h_T \circ A) d\nu \pm \varrho(\text{Supp } A)$$

where ν, ϱ are classical measures on \mathcal{B} .

Particular case of strict Frank t-norms:

$$\mu(A) = \int A d\nu \pm \varrho(\text{Supp } A)$$

Measures on full tribes, strict nearly Frank t-norms

Theorem: [Butnariu, Klement; Mesiar, MN] Each measure is of the form

$$\mu(A) = \int (h_T \circ A) d\nu \pm \varrho(\text{Supp } A)$$

where ν, ϱ are classical measures on \mathcal{B} .

Particular case of strict Frank t-norms:

$$\mu(A) = \int A d\nu \pm \varrho(\text{Supp } A)$$

measures	tribes	T nearly Frank (with h_T)	T not nearly Frank
regular	full	$\int (h_T \circ A) d\nu$	0
all	full	$\int (h_T \circ A) d\nu \pm \varrho(\text{Supp } A)$	$\varrho(\text{Supp } A)$

Examples of measures on the full tribe of constants $([0, 1], T)$, T **strict nearly Frank**

Examples of measures on the full tribe of constants $([0, 1], T)$, T **strict nearly Frank**

Example:

$$\kappa(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It is a support measure on a singleton.

It is **monotone**, but **not regular**.

Examples of measures on the full tribe of constants $([0, 1], T)$, T **strict nearly Frank**

Example:

$$\kappa(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It is a support measure on a singleton.

It is **monotone**, but **not regular**.

Example:

$$\mu(x) = \begin{cases} 1 - \frac{x}{2} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It is a measure which is a linear combination of a support measure κ and a regular measure $\nu = id$, $\mu = \kappa - \frac{1}{2} id$

It is **neither monotone nor regular**.

Charges

Alternative notion:

Signed measure (charge) on a tribe (\mathcal{T}, T) :

$\mu: \mathcal{T} \rightarrow \mathbb{R}$ s.t.

- ◆ $\mu(0) = 0$
- ◆ $\mu(A \dot{\cup} B) = \mu(A) + \mu(B) - \mu(A \cap B)$
- ◆ $A_n \nearrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$

Charges

Alternative notion:

Signed measure (charge) on a tribe (\mathcal{T}, T) :

$\mu: \mathcal{T} \rightarrow \mathbb{R}$ s.t.

- ◆ $\mu(0) = 0$
- ◆ $\mu(A \dot{\cup} B) = \mu(A) + \mu(B) - \mu(A \cap B)$
- ◆ $A_n \nearrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$

Moreover, for a **regular signed measure (regular charge)** we require

- ◆ $A_n \searrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$
-

Tribes which are **not full**

Tribes which are **not full**

They exist

Tribes which are **not full**

They exist

Example: Boolean tribes

Tribes which are **not full**

They exist

Example: Boolean tribes

Example:

$\mathcal{T} = \{A \in [0, 1]^{\mathbb{R}} \mid A \text{ is Borel-measurable, } A(x) \in \{0, 1\} \text{ almost everywhere}\}$
(T an arbitrary measurable t-norm)

Tribes which are **not full**

They exist

Example: Boolean tribes

Example:

$\mathcal{T} = \{A \in [0, 1]^{\mathbb{R}} \mid A \text{ is Borel-measurable, } A(x) \in \{0, 1\} \text{ almost everywhere}\}$
 (T an arbitrary measurable t-norm)

Example: [Butnariu, Klement; Mesiar; MN]

\mathcal{B} ... a σ -algebra of subsets of X

Δ ... a σ -ideal in \mathcal{B}

$\mathcal{T} = \{A \in [0, 1]^X \mid A \text{ is } \mathcal{B}\text{-measurable, } A^{-1}(]0, 1[) \in \Delta\}$

(\mathcal{T}, T) is a tribe called a **weakly full tribe** (**weakly generated tribe**)

Tribes which are weakly full

There are many strict t-norms T such that all tribes (\mathcal{T}, T) are weakly full.

Tribes which are weakly full

There are many strict t-norms T such that all tribes (\mathcal{T}, T) are weakly full.

These are strict **sufficient** t-norms, i.e., t-norms which – with the standard negation and limits of monotone sequences approximate **sufficient sets of fuzzy logical operations** “as much as possible” [Butnariu, Klement, Mesiar, MN]

Tribes which are weakly full

There are many strict t-norms T such that all tribes (\mathcal{T}, T) are weakly full.

These are strict **sufficient** t-norms, i.e., t-norms which – with the standard negation and limits of monotone sequences approximate **sufficient sets of fuzzy logical operations** “as much as possible” [Butnariu, Klement, Mesiar, MN]

Example: Strict t-norms from the following families are sufficient: Aczél–Alsina, Frank, Hamacher*, the eighth Mizumoto, the tenth Mizumoto, Schweizer–Sklar*, the third Schweizer, etc.

* except for one value of the parameter

Tribes which are **not weakly full**

Tribes which are **not weakly full**

They exist (even for **strict** t-norms)

Tribes which are **not weakly full**

They exist (even for **strict** t-norms)

Example: Hamacher product, $T_0^{\mathbf{H}}$:

$$T_0^{\mathbf{H}}(x, y) = \begin{cases} 0 & \text{if } x = y = 0, \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

Let $X = \{x, y\}$. There is a distance d on $[0, 1]$ and $c \in \mathbb{R}$ such that $(\mathcal{T}, T_0^{\mathbf{H}})$ is a tribe, where $\mathcal{T} = \{0, 1\} \cup \{A \in]0, 1[^X \mid d(A(x), A(y)) \leq c\}$

Tribes which are **not weakly full**

They exist (even for **strict** t-norms)

Example: Hamacher product, $T_0^{\mathbf{H}}$:

$$T_0^{\mathbf{H}}(x, y) = \begin{cases} 0 & \text{if } x = y = 0, \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

Let $X = \{x, y\}$. There is a distance d on $[0, 1]$ and $c \in \mathbb{R}$ such that $(\mathcal{T}, T_0^{\mathbf{H}})$ is a tribe, where $\mathcal{T} = \{0, 1\} \cup \{A \in]0, 1[^X \mid d(A(x), A(y)) \leq c\}$

Further counterexamples:

T-norms T obtained from the Hamacher product by the formula

$$T(x, y) = h_T^{-1}(T_0^{\mathbf{H}}(h_T(x), h_T(y))),$$

where h_T is an order automorphism of $[0, 1]$ which commutes with \neg

Further counterexamples

This way, we obtain

Further counterexamples

This way, we obtain

- ◆ strict Dombi t-norms ($\lambda \in]0, \infty[$)

$$x \underset{D_\lambda}{\wedge} y = \frac{1}{\left(\left(\frac{1}{x} - 1 \right)^\lambda + \left(\frac{1}{y} - 1 \right)^\lambda \right)^{\frac{1}{\lambda}} + 1}$$

Further counterexamples

This way, we obtain

- ◆ strict Dombi t-norms ($\lambda \in]0, \infty[$)

$$x \underset{D_\lambda}{\wedge} y = \frac{1}{\left(\left(\frac{1}{x} - 1 \right)^\lambda + \left(\frac{1}{y} - 1 \right)^\lambda \right)^{\frac{1}{\lambda}} + 1}$$

- ◆ the first Mizumoto t-norm

$$x \underset{M_1}{\wedge} y = \frac{2}{\pi} \operatorname{arccot} \left(\cot \frac{\pi}{2} x + \cot \frac{\pi}{2} y \right)$$

Further counterexamples

This way, we obtain

- ◆ strict Dombi t-norms ($\lambda \in]0, \infty[$)

$$x \underset{D_\lambda}{\wedge} y = \frac{1}{\left(\left(\frac{1}{x} - 1 \right)^\lambda + \left(\frac{1}{y} - 1 \right)^\lambda \right)^{\frac{1}{\lambda}} + 1}$$

- ◆ the first Mizumoto t-norm

$$x \underset{M1}{\wedge} y = \frac{2}{\pi} \operatorname{arccot} \left(\cot \frac{\pi}{2} x + \cot \frac{\pi}{2} y \right)$$

Theorem: All other strict t-norms **found in the literature** are sufficient.

Tribes which are **not weakly full**

Open problem: Characterize tribes for non-sufficient t-norms.

Tribes which are **not weakly full**

Open problem: Characterize tribes for non-sufficient t-norms.

Although we do not know a characterization of tribes, we can characterize **regular** measures on them.

Tribes which are **not weakly full**

Open problem: Characterize tribes for non-sufficient t-norms.

Although we do not know a characterization of tribes, we can characterize **regular** measures on them.

Extreme cases:

Full tribes: **no** non-zero regular measures.

Tribes which are **not weakly full**

Open problem: Characterize tribes for non-sufficient t-norms.

Although we do not know a characterization of tribes, we can characterize **regular** measures on them.

Extreme cases:

Full tribes: **no** non-zero regular measures.

Boolean tribes: **many** non-zero regular measures.

Tribes which are **not weakly full**

Open problem: Characterize tribes for non-sufficient t-norms.

Although we do not know a characterization of tribes, we can characterize **regular** measures on them.

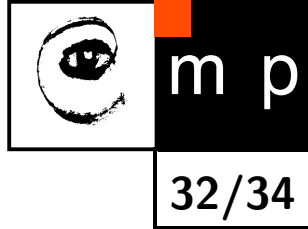
Extreme cases:

Full tribes: **no** non-zero regular measures.

Boolean tribes: **many** non-zero regular measures.

General case: something between.

Regular measures on tribes



$\Delta_{\mathcal{T}} = \{A^{-1}(]0, 1[) \mid A \in \mathcal{T}\}$ is a σ -ideal in \mathcal{B}

Regular measures on tribes

$\Delta_{\mathcal{T}} = \{A^{-1}(]0, 1[) \mid A \in \mathcal{T}\}$ is a σ -ideal in \mathcal{B}

Theorem: If \mathcal{T} is strict and **not nearly Frank**, then each measure is of the form

$$\mu(A) = \varrho(\text{Supp } A)$$

where ϱ is a (classical) measure on \mathcal{B} which **vanishes at** $\Delta_{\mathcal{T}}$

Regular measures on tribes

$\Delta_{\mathcal{T}} = \{A^{-1}(]0, 1[) \mid A \in \mathcal{T}\}$ is a σ -ideal in \mathcal{B}

Theorem: If \mathcal{T} is strict and **not nearly Frank**, then each measure is of the form

$$\mu(A) = \varrho(\text{Supp } A)$$

where ϱ is a (classical) measure on \mathcal{B} which **vanishes at** $\Delta_{\mathcal{T}}$

For strict nearly Frank t-norms the tribes are weakly full and the characterization is as before.

Regular measures on tribes

$\Delta_{\mathcal{T}} = \{A^{-1}(]0, 1[) \mid A \in \mathcal{T}\}$ is a σ -ideal in \mathcal{B}

Theorem: If T is strict and **not nearly Frank**, then each measure is of the form

$$\mu(A) = \varrho(\text{Supp } A)$$

where ϱ is a (classical) measure on \mathcal{B} which **vanishes at** $\Delta_{\mathcal{T}}$

For strict nearly Frank t-norms the tribes are weakly full and the characterization is as before.

measures	tribes	T nearly Frank (with h_T)	T not nearly Frank
regular	full	$\int (h_T \circ A) d\nu$	0
all	full	$\int (h_T \circ A) d\nu \pm \varrho(\text{Supp } A)$	$\varrho(\text{Supp } A)$
regular	general	$\int (h_T \circ A) d\nu$	$\varrho(\text{Supp } A)$ *

* where $\varrho \upharpoonright \Delta_{\mathcal{T}} = 0$; then $\varrho(\text{Supp } A) = \varrho(A^{-1}(1)) = \int (h_T \circ A) d\varrho$
 (h_T arbitrary)

Measures on tribes

Measures on tribes

Characterization of measures is known only for weakly full tribes (e.g., if T is sufficient).

Measures on tribes

Characterization of measures is known only for weakly full tribes (e.g., if T is sufficient).

Then measures are as before [Barbieri, H. Weber, MN]:

measures	tribes	T nearly Frank (with h_T)	T not nearly Frank
regular	full	$\int (h_T \circ A) d\nu$	0
all	full	$\int (h_T \circ A) d\nu \pm \varrho(\text{Supp } A)$	$\varrho(\text{Supp } A)$
regular	general	$\int (h_T \circ A) d\nu$	$\varrho(\text{Supp } A)^*$
all	general	$\int (h_T \circ A) d\nu \pm \varrho(\text{Supp } A)$	$\varrho(\text{Supp } A)^{**}$

* where $\varrho \upharpoonright \Delta_T = 0$; then $\varrho(\text{Supp } A) = \varrho(A^{-1}(1)) = \int (h_T \circ A) d\varrho$ (h_T arbitrary)

** only for weakly full tribes

Measures on tribes

Characterization of measures is known only for weakly full tribes (e.g., if T is sufficient).

Then measures are as before [Barbieri, H. Weber, MN]:

measures	tribes	T nearly Frank (with h_T)	T not nearly Frank
regular	full	$\int (h_T \circ A) d\nu$	0
all	full	$\int (h_T \circ A) d\nu \pm \varrho(\text{Supp } A)$	$\varrho(\text{Supp } A)$
regular	general	$\int (h_T \circ A) d\nu$	$\varrho(\text{Supp } A)^*$
all	general	$\int (h_T \circ A) d\nu \pm \varrho(\text{Supp } A)$	$\varrho(\text{Supp } A)^{**}$

* where $\varrho \upharpoonright \Delta_T = 0$; then $\varrho(\text{Supp } A) = \varrho(A^{-1}(1)) = \int (h_T \circ A) d\varrho$ (h_T arbitrary)

** only for weakly full tribes

Open problem: Characterize measures on tribes which are not weakly full.

Measures on tribes

Characterization of measures is known only for weakly full tribes (e.g., if T is sufficient).

Then measures are as before [Barbieri, H. Weber, MN]:

measures	tribes	T nearly Frank (with h_T)	T not nearly Frank
regular	full	$\int (h_T \circ A) d\nu$	0
all	full	$\int (h_T \circ A) d\nu \pm \varrho(\text{Supp } A)$	$\varrho(\text{Supp } A)$
regular	general	$\int (h_T \circ A) d\nu$	$\varrho(\text{Supp } A)^*$
all	general	$\int (h_T \circ A) d\nu \pm \varrho(\text{Supp } A)$	$\varrho(\text{Supp } A)^{**}$

* where $\varrho \upharpoonright \Delta_T = 0$; then $\varrho(\text{Supp } A) = \varrho(A^{-1}(1)) = \int (h_T \circ A) d\varrho$ (h_T arbitrary)

** only for weakly full tribes

Open problem: Characterize measures on tribes which are not weakly full.

For **regular** measures, the characterization is known in full generality.

CONCLUSION

We have a reasonable generalization of measure theory for tribes of fuzzy sets.

(There are analogues of Jordan and Hahn decomposition, Lyapunov theorem, etc.)

CONCLUSION

We have a reasonable generalization of measure theory for tribes of fuzzy sets.

(There are analogues of Jordan and Hahn decomposition, Lyapunov theorem, etc.)

In contrast to preceding work, most of the results do not need any assumptions on the structure of the tribe.

CONCLUSION

We have a reasonable generalization of measure theory for tribes of fuzzy sets.

(There are analogues of Jordan and Hahn decomposition, Lyapunov theorem, etc.)

In contrast to preceding work, most of the results do not need any assumptions on the structure of the tribe.

Regular measures seem to be a reasonable alternative to the original definition by Butnariu and Klement.

CONCLUSION

We have a reasonable generalization of measure theory for tribes of fuzzy sets.

(There are analogues of Jordan and Hahn decomposition, Lyapunov theorem, etc.)

In contrast to preceding work, most of the results do not need any assumptions on the structure of the tribe.

Regular measures seem to be a reasonable alternative to the original definition by Butnariu and Klement.

The characterization of **charges** is simpler than that of **measures**.

CONCLUSION

We have a reasonable generalization of measure theory for tribes of fuzzy sets.

(There are analogues of Jordan and Hahn decomposition, Lyapunov theorem, etc.)

In contrast to preceding work, most of the results do not need any assumptions on the structure of the tribe.

Regular measures seem to be a reasonable alternative to the original definition by Butnariu and Klement.

The characterization of **charges** is simpler than that of **measures**.

Frank (more exactly, nearly Frank) t-norms play a prominent role in the characterization of measures.