Probability and conditional probability on tribes of fuzzy sets

Mirko Navara
Center for Machine Perception
Department of Cybernetics
Faculty of Electrical Engineering
Czech Technical University
CZ-166 27 Praha, Czech Republic
http://cmp.felk.cvut.cz/~navara
Zadeh’s idea of measures of fuzzy sets

\[ P(A) = \int A \, d\nu \]

where \( \nu \) is a classical probability measure

\( P \) is an integral measure.

The contribution of each element is weighted by its membership degree.
Zadeh’s idea of measures of fuzzy sets

\[ P(A) = \int A \, d\nu \]

where \( \nu \) is a classical probability measure

\( P \) is an \textbf{integral measure}.

The contribution of each element is weighted by its membership degree

It requires to specify:
Zadeh’s idea of measures of fuzzy sets

\[ P(A) = \int A \, d\nu \]

where \( \nu \) is a classical probability measure

\( P \) is an **integral measure**.

The contribution of each element is weighted by its membership degree

It requires to specify:

- the underlying \( \sigma \)-**algebra** w.r.t. which \( A \) has to be measurable,
Zadeh’s idea of measures of fuzzy sets

\[ P(A) = \int A \, d\nu \]

where \( \nu \) is a classical probability measure

\( P \) is an **integral measure**.

The contribution of each element is weighted by its membership degree

It requires to specify:

- the underlying \( \sigma \)-**algebra** w.r.t. which \( A \) has to be measurable,
- the **classical measure** \( \nu \),
Zadeh’s idea of measures of fuzzy sets

\[ P(A) = \int A \, d\nu \]

where \( \nu \) is a classical probability measure

\( P \) is an **integral measure**.

The contribution of each element is weighted by its membership degree

It requires to specify:

- the underlying \( \sigma \)-algebra w.r.t. which \( A \) has to be measurable,
- the **classical measure** \( \nu \),
- the **domain** of the measure \( P \).
Axiomatic approach 1A: MV-algebras

Here only • $\sigma$-complete,
Axiomatic approach 1A: MV-algebras

Here only • \(\sigma\)-complete,

• representable as collections of fuzzy sets
Axiomatic approach 1A: MV-algebras

Here only

- $\sigma$-complete,
- representable as collections of fuzzy sets

$\Łukasiewicz tribes \equiv \text{systems } T \text{ of fuzzy sets s.t.}$

(T1) $0 \in T$,

(T2) $A \in T \Rightarrow A' = 1 - A \in T$,

(T3) $A, B \in T \Rightarrow A \odot B \in T$,

(T4) $(A_n)_{n \in \mathbb{N}} \in T^\mathbb{N}, A_n \nearrow A \Rightarrow A \in T$,

where $\odot$ is the Łukasiewicz t-norm and $\oplus$ is the Łukasiewicz t-conorm.
Axiomatic approach 1A: MV-algebras

Here only
• σ-complete,
• representable as collections of fuzzy sets

= Łukasiewicz tribes = systems $\mathcal{T}$ of fuzzy sets s.t.

(T1) $0 \in \mathcal{T},$

(T2) $A \in \mathcal{T} \Rightarrow A' = 1 - A \in \mathcal{T},$

(T3) $A, B \in \mathcal{T} \Rightarrow A \circ B \in \mathcal{T},$

(T4) $(A_n)_{n \in \mathbb{N}} \in \mathcal{T}^{\mathbb{N}}, A_n \nearrow A \Rightarrow A \in \mathcal{T},$

where $\circ$ is the Łukasiewicz t-norm and $\oplus$ is the Łukasiewicz t-conorm.

Equivalently, (T3) and (T4) may be replaced by

(TM3+) $(A_n)_{n \in \mathbb{N}} \in \mathcal{T}^{\mathbb{N}} \Rightarrow \bigcirc_n A_n \in \mathcal{T}.$
1A: Measures on MV-algebras

Probability measure (≡ state) $P: \mathcal{T} \rightarrow [0, 1]$ s.t.

(M1) $P(1) = 1$,

(M2) $A, B \in \mathcal{T}, \ A \circ B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B)$, (additivity)

(M3) $(A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, \ A_n \nearrow A \Rightarrow P(A_n) \rightarrow P(A)$. 
1A: Measures on MV-algebras

**Probability measure** (=state) $P: \mathcal{T} \rightarrow [0, 1]$ s.t.

(M1) $P(1) = 1,$

(M2) $A, B \in \mathcal{T}, \ A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B),$ (additivity)

(M3) $(A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, \ A_n \nearrow A \Rightarrow P(A_n) \rightarrow P(A)$.

Crisp elements of $\mathcal{T}$ determine a $\sigma$-algebra $\mathcal{B}(\mathcal{T})$.

**Theorem:** All elements of $\mathcal{T}$ are $\mathcal{B}(\mathcal{T})$-measurable.
1A: Measures on MV-algebras

**Probability measure** (=state) $P : \mathcal{T} \to [0, 1]$ s.t.

(M1) $P(1) = 1$,

(M2) $A, B \in \mathcal{T}$, $A \otimes B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B)$, *(additivity)*

(M3) $(A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}$, $A_n ↠ A \Rightarrow P(A_n) ↠ P(A)$.

Crisp elements of $\mathcal{T}$ determine a $\sigma$-algebra $\mathcal{B}(\mathcal{T})$.

**Theorem**: All elements of $\mathcal{T}$ are $\mathcal{B}(\mathcal{T})$-measurable.

Each state $P$ is an integral measure,

$$P(A) = \int A \, d\nu,$$

where $\nu = P \upharpoonright \mathcal{B}(\mathcal{T})$ is a classical probability measure.
Axiomatic approach 1B: MV-algebras with product

[Mundici] MV-algebras admit joint refinements of partitions of unity.
Axiomatic approach 1B: MV-algebras with product

[Mundici] MV-algebras admit joint refinements of partitions of unity.

However, there is no canonical formula for a joint refinement,
Axiomatic approach 1B: MV-algebras with product

[Mundici] MV-algebras admit joint refinements of partitions of unity.

However, there is no canonical formula for a joint refinement,
there is no coarsest (or otherwise distinguished) joint refinement.
Axiomatic approach 1B: MV-algebras with product

[Mundici] MV-algebras admit joint refinements of partitions of unity.

However, there is no canonical formula for a joint refinement,

there is no coarsest (or otherwise distinguished) joint refinement.

To introduce joint distributions (independence, limit theorems, laws of large numbers, ...), we need more:
Axiomatic approach 1B: MV-algebras with product

[Mundici] MV-algebras admit joint refinements of partitions of unity.

However, there is no canonical formula for a joint refinement, there is no coarsest (or otherwise distinguished) joint refinement.

To introduce joint distributions (independence, limit theorems, laws of large numbers, ...), we need more:

[Riečan & Mundici] An MV-algebra with **product** \( \cdot : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \) s.t.

(P1) \( 1 \cdot A = A \),

(P2) \( A \cdot (B \ominus C) = (A \cdot B) \ominus (A \cdot C) \).
Axiomatic approach 1B: **MV-algebras with product**

[Mundici] MV-algebras admit joint refinements of partitions of unity.

However, there is no canonical formula for a joint refinement,

there is no coarsest (or otherwise distinguished) joint refinement.

To introduce joint distributions (independence, limit theorems, laws of large numbers, ...), we need more:

[Riečan & Mundici] An MV-algebra with **product** \( \cdot : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \) s.t.

\[
\begin{align*}
(P1) \quad & 1 \cdot A = A, \\
(P2) \quad & A \cdot (B \ominus C) = (A \cdot B) \ominus (A \cdot C).
\end{align*}
\]

**Theorem:** \( (A \cdot B)(x) = A(x) \cdot B(x) \)
Axiomatic approach 1B: **MV-algebras with product**

[Mundici] MV-algebras admit joint refinements of partitions of unity.

However, there is no canonical formula for a joint refinement, there is no coarsest (or otherwise distinguished) joint refinement.

To introduce joint distributions (independence, limit theorems, laws of large numbers, ...), we need more:

[Riečan & Mundici] An MV-algebra with product $\cdot : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ s.t.

(P1) $1 \cdot A = A$,

(P2) $A \cdot (B \ominus C) = (A \cdot B) \ominus (A \cdot C)$.

**Theorem:** $(A \cdot B)(x) = A(x) \cdot B(x)$

States remain the same.
Axiomatic approach 2A: Tribes

[Höhle, Klement, Butnariu, Mesiar, MN, S. Weber, H. Weber, Barbieri, ...]
Axiomatic approach 2A: Tribes

[Höhle, Klement, Butnariu, Mesiar, MN, S. Weber, H. Weber, Barbieri, ...]

Fix a t-norm $\odot$ and its dual t-conorm $\oplus$. 
Axiomatic approach 2A: Tribes

[Höhle, Klement, Butnariu, Mesiar, MN, S. Weber, H. Weber, Barbieri, ...]

Fix a t-norm $\odot$ and its dual t-conorm $\oplus$.

**Tribe** = system $\mathcal{T}$ of fuzzy sets s.t.

(T1) $0 \in \mathcal{T}$,

(T2) $A \in \mathcal{T} \Rightarrow A' = 1 - A \in \mathcal{T}$,

(T3) $A, B \in \mathcal{T} \Rightarrow A \odot B \in \mathcal{T}$,

(T4) $(A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}$, $A_n \nearrow A \Rightarrow A \in \mathcal{T}$. 
Axiomatic approach 2A: Tribes

[Höhle, Klement, Butnariu, Mesiar, MN, S. Weber, H. Weber, Barbieri, ...]

Fix a t-norm $\odot$ and its dual t-conorm $\oplus$.

Tribe $= \text{system } \mathcal{T} \text{ of fuzzy sets s.t.}$

(T1) $0 \in \mathcal{T}$,

(T2) $A \in \mathcal{T} \Rightarrow A' = 1 - A \in \mathcal{T}$,

(T3) $A, B \in \mathcal{T} \Rightarrow A \odot B \in \mathcal{T}$,

(T4) $(A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, \ A_n \uparrow A \Rightarrow A \in \mathcal{T}$.

Originally [Butnariu, Klement], (T3) and (T4) were replaced by

(T3+) $(A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N} \Rightarrow \bigodot_n A_n \in \mathcal{T},$

which is weaker. Nevertheless, it is equivalent in the cases they studied.
2A: Measures on tribes

**Probability measure** $P : \mathcal{T} \to [0, 1]$ s.t.

(M1) $P(1) = 1,$

(M2*) $A, B \in \mathcal{T} \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B),$

(M3) $(A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, \ A_n \nearrow A \Rightarrow P(A_n) \to P(A),$
2A: Measures on tribes

**Probability measure** $P : \mathcal{T} \rightarrow [0, 1]$ s.t.

(M1) $P(1) = 1,$

(M2$^*$) $A, B \in \mathcal{T} \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B),$

(M3) $(A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, \ A_n \uparrow A \Rightarrow P(A_n) \rightarrow P(A),$

(M2$^*$) is the **valuation property**, stronger than additivity

(M2) $A, B \in \mathcal{T}, \ A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B),$ which, for t-norms without zero divisors (minimum, product, ...) applies only to fuzzy sets with disjoint supports, $(\text{Supp}(A) = A^{-1}([0, 1])).$
The interplay of valuation property $(M2^*)$ and additivity $(M2)$

$(M2^*) \ A, B \in \mathcal{T} \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B),$

$(M2) \ A, B \in \mathcal{T}, \ A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B).$
The interplay of valuation property \((M2^*)\) and additivity \((M2)\)

\[(M2^*)\] \(A, B \in \mathcal{T} \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B),\]

\[(M2)\] \(A, B \in \mathcal{T}, A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B).\]

- For the standard operations, \(\odot = \min\), \(\oplus = \max\), both are as weak.
The interplay of valuation property (M2*) and additivity (M2)

(M2*) $A, B \in \mathcal{T} \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B),$

(M2) $A, B \in \mathcal{T}, \ A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B).$

• For the standard operations, $\odot = \min, \ \oplus = \max$, both are as weak.

• For the Łukasiewicz operations, both are as strong (case 1A).
The interplay of valuation property \((M2^*)\) and additivity \((M2)\)

\[(M2^*)\] \(A, B \in \mathcal{T} \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B),\]

\[(M2)\] \(A, B \in \mathcal{T}, A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B).\]

- For the **standard** operations, \(\odot = \min\), \(\oplus = \max\), both are as **weak**.
- For the **Łukasiewicz** operations, both are as **strong** (case 1A).
- For the **strict** operations, \((M2)\) is as **weak** as for **standard** operations,
The interplay of valuation property \((M2^*)\) and additivity \((M2)\)

\((M2^*)\) \(A, B \in T \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B),\)

\((M2)\) \(A, B \in T, A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B).\)

- For the \textit{standard} operations, \(\odot = \text{min}, \oplus = \text{max}, \text{both} \) are as \textit{weak}.
- For the \textit{Łukasiewicz} operations, \textit{both} are as \textit{strong} (case 1A).
- For the \textit{strict} operations, \((M2)\) is as \textit{weak} as for \textit{standard} operations, 
  \((M2^*)\) \textit{may be} as \textit{weak} as for \textit{standard} operations, but also
The interplay of valuation property \((M2^*)\) and additivity \((M2)\)

\[(M2^*) \quad A, B \in T \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B),\]

\[(M2) \quad A, B \in T, \quad A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B).\]

- For the **standard** operations, \(\odot = \min\), \(\oplus = \max\), both are as **weak**.
- For the Łukasiewicz operations, both are as **strong** (case 1A).
- For the **strict** operations, \((M2)\) is as **weak** as for standard operations,

\[(M2^*)\] may be as **weak** as for standard operations, but also

\[(M2^*)\] may be as **strong** as for Łukasiewicz operations
(depending on the t-norm).
The interplay of valuation property $(M2^*)$ and additivity $(M2)$

$$(M2^*) \quad A, B \in T \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B),$$

$$(M2) \quad A, B \in T, \ A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B).$$

• For the **standard** operations, $\odot = \min$, $\oplus = \max$, **both** are as **weak**.

• For the Łukasiewicz operations, **both** are as **strong** (case 1A).

• For the **strict** operations, $(M2)$ is as **weak** as for **standard** operations,

$$(M2^*) \text{ may be as weak as for standard operations, but also}$$

$$(M2^*) \text{ may be as strong as for Łukasiewicz operations (depending on the t-norm).}$$

From now on, we restrict only to **strict** t-norms and their dual t-conorms.
2A: Characterization of measures on tribes

Crisp elements of $\mathcal{T}$ determine a $\sigma$-algebra $\mathcal{B}(\mathcal{T})$. 

2A: Characterization of measures on tribes

Crisp elements of $\mathcal{T}$ determine a $\sigma$-algebra $\mathcal{B}(\mathcal{T})$.

There are always support measures of the form

$$P(A) = \nu(\text{Supp}(A)),$$

where $\nu = P \mid \mathcal{B}(\mathcal{T})$ is a classical probability measure.
2A: Characterization of measures on tribes

Crisp elements of $\mathcal{T}$ determine a $\sigma$-algebra $\mathcal{B}(\mathcal{T})$.

There are always support measures of the form

$$P(A) = \nu(\text{Supp} (A)),$$

where $\nu = P \upharpoonright \mathcal{B}(\mathcal{T})$ is a classical probability measure.

Only some strict t-norms admit also integral measures of the form

$$P(A) = \int A \, d\nu$$

and convex combinations of support and integral measures.
2A: Characterization of measures on tribes

Crisp elements of $\mathcal{T}$ determine a $\sigma$-algebra $\mathcal{B}(\mathcal{T})$.

There are always support measures of the form

$$P(A) = \nu(\text{Supp}(A)),$$

where $\nu = P | \mathcal{B}(\mathcal{T})$ is a classical probability measure.

Only some strict t-norms admit also integral measures of the form

$$P(A) = \int A \, d\nu$$

and convex combinations of support and integral measures.

This happens for the product and for strict Frank t-norms

$$x \circ y = \log_{\lambda} \left( 1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1} \right) \quad \text{for } \lambda \in ]0, \infty[ \setminus \{1\}$$
Nearly Frank t-norms

We change the membership degrees by an increasing bijection $h: [0, 1] \rightarrow [0, 1]$ which **commutes with the standard negation**, \[ \forall \alpha \in [0, 1] : h(1 - \alpha) = 1 - h(\alpha). \]
Nearly Frank t-norms

We change the membership degrees by an increasing bijection $h: [0, 1] \rightarrow [0, 1]$ which *commutes with the standard negation*, $\forall \alpha \in [0, 1]: h(1 - \alpha) = 1 - h(\alpha)$.

- The Frank t-norm $\circ$ changes to a **nearly Frank t-norm**

\[
x \circ_h y = h^{-1}(h(x) \circ h(y)).
\]
**Nearly Frank t-norms**

We change the membership degrees by an increasing bijection 
\( h: [0, 1] \to [0, 1] \) which **commutes with the standard negation**, 
\( \forall \alpha \in [0, 1]: h(1 - \alpha) = 1 - h(\alpha) \).

- The Frank t-norm \( \odot \) changes to a **nearly Frank t-norm**

\[
x \odot_h y = h^{-1}(h(x) \odot h(y)).
\]

- The tribe is preserved.
Nearly Frank t-norms

We change the membership degrees by an increasing bijection $h: [0, 1] \rightarrow [0, 1]$ which commutes with the standard negation, $\forall \alpha \in [0, 1]: h(1 - \alpha) = 1 - h(\alpha)$.

- The Frank t-norm $\circ$ changes to a nearly Frank t-norm
  \[ x \circ_h y = h^{-1}(h(x) \circ h(y)) \].

- The tribe is preserved.
- Support measures remain unchanged.
Nearly Frank t-norms

We change the membership degrees by an increasing bijection $h: [0, 1] \rightarrow [0, 1]$ which **commutes with the standard negation**, $\forall \alpha \in [0, 1]: h(1 - \alpha) = 1 - h(\alpha)$.

- The Frank t-norm $\circ$ changes to a **nearly Frank t-norm**

  $$x \circ_h y = h^{-1}(h(x) \circ h(y)).$$

- The tribe is preserved.
- Support measures remain unchanged.
- Integral measures become

  $$P(A) = \int h \circ A \, dv.$$
Nearly Frank t-norms

We change the membership degrees by an increasing bijection $h: [0, 1] \rightarrow [0, 1]$ which commute with the standard negation, \( \forall \alpha \in [0, 1]: h(1 - \alpha) = 1 - h(\alpha) \).

- The Frank t-norm \( \circ \) changes to a nearly Frank t-norm
  \[
  x \circ_h y = h^{-1}(h(x) \circ h(y)) .
  \]

- The tribe is preserved.
- Support measures remain unchanged.
- Integral measures become
  \[
  P(A) = \int h \circ A \, d\nu .
  \]

Up to the change of scale $h$, everything remains analogous.
Nearly Frank $t$-norms

**Theorem:** If the $t$-norm is *not nearly Frank*, there are only support measures.
Nearly Frank t-norms

**Theorem**: If the t-norm is not nearly Frank, there are only support measures.

**Objection**: Support measures depend only on the support, they do not distinguish between non-zero membership degrees.
Nearly Frank t-norms

**Theorem:** If the t-norm is not nearly Frank, there are only support measures.

**Objection:** Support measures depend only on the support, they do not distinguish between non-zero membership degrees.

This can hardly be motivated.
Axiomatic approach 2B: Frank tribes

[Klement, Butnariu, Mesiar, MN, ...]

⊙ is the **product** or a **strict Frank** t-norm

(M1) \( P(1) = 1, \)

(M2*) \( A, B \in \mathcal{T} \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B), \)

(M3) \( (A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, A_n \nearrow A \Rightarrow P(A_n) \to P(A), \)

(M4) \( (A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, A_n \searrow A \Rightarrow P(A_n) \to P(A). \) (**σ-order continuity**)
Axiomatic approach 2B: Frank tribes

[Klement, Butnariu, Mesiar, MN, ...]

⊙ is the **product** or a **strict Frank** **t-norm**

(M1) \( P(1) = 1 \),

(M2*) \( A, B \in \mathcal{T} \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B) \),

(M3) \( (A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, A_n \uparrow A \Rightarrow P(A_n) \to P(A) \),

(M4) \( (A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, A_n \downarrow A \Rightarrow P(A_n) \to P(A) \). (**\(\sigma\)-order continuity)**

(M4) is fulfilled and unnecessary in Boolean \(\sigma\)-algebras and 1A, 1B,
Axiomatic approach 2B: Frank tribes

[Klement, Butnariu, Mesiar, MN, ...]

⊙ is the **product** or a **strict Frank** t-norm

(M1) \( P(1) = 1, \)

(M2\(^*\)) \( A, B \in \mathcal{T} \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B), \)

(M3) \( (A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, A_n \uparrow A \Rightarrow P(A_n) \to P(A), \)

(M4) \( (A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, A_n \downarrow A \Rightarrow P(A_n) \to P(A). \) (\( \sigma \)-order continuity)

(M4) is fulfilled and unnecessary in Boolean \( \sigma \)-algebras and 1A, 1B, not satisfied in general tribes (case 2A).
Axiomatic approach 2B: Frank tribes

[Klement, Butnariu, Mesiar, MN, ...]

⊙ is the **product** or a **strict Frank** t-norm

(M1) \( P(1) = 1, \)

(M2\(^*\)) \( A, B \in \mathcal{T} \Rightarrow P(A \oplus B) = P(A) + P(B) - P(A \odot B), \)

(M3) \( (A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, A_n \nearrow A \Rightarrow P(A_n) \to P(A), \)

(M4) \( (A_n)_{n \in \mathbb{N}} \in \mathcal{T}^\mathbb{N}, A_n \searrow A \Rightarrow P(A_n) \to P(A). \) (**σ-order continuity**)  

(M4) is fulfilled and unnecessary in Boolean \( σ \)-algebras and 1A, 1B, not satisfied in general tribes (case 2A).

(M4) does not admit support measures.
2B: Characterization of $\sigma$-order continuous measures on Frank tribes
2B: Characterization of $\sigma$-order continuous measures on Frank tribes

Theorem:
2B: Characterization of $\sigma$-order continuous measures on Frank tribes

Theorem:

The tribe $\mathcal{T}$ is also a Łukasiewicz tribe with product.
Theorem:

The tribe $\mathcal{T}$ is also a Łukasiewicz tribe with product.

All elements of $\mathcal{T}$ are $\mathcal{B}(\mathcal{T})$-measurable.
Theorem:

The tribe $\mathcal{T}$ is also a Łukasiewicz tribe with product.

All elements of $\mathcal{T}$ are $\mathcal{B}(\mathcal{T})$-measurable.

All $\sigma$-order continuous probability measures are integral measures of the form

$$P(A) = \int A \, d\nu,$$

where $\nu = P \mid \mathcal{B}(\mathcal{T})$ is a classical probability measure.
## 1B and 2B: Comparison

<table>
<thead>
<tr>
<th>1B</th>
<th>2B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Łukasiewicz tribe</strong></td>
<td><strong>product or strict Frank tribe</strong></td>
</tr>
<tr>
<td><strong>product</strong> as an additional operation</td>
<td><strong>Łukasiewicz operations derived</strong></td>
</tr>
<tr>
<td><strong>(M2)</strong> ( A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B) )</td>
<td><em><em>(M2</em>)</em>* ( P(A \oplus B) = P(A) + P(B) - P(A \odot B) )</td>
</tr>
<tr>
<td><strong>(M4) satisfied</strong></td>
<td><strong>(M4) required</strong></td>
</tr>
<tr>
<td>( A_n \searrow A \Rightarrow P(A_n) \rightarrow P(A) )</td>
<td></td>
</tr>
</tbody>
</table>
## 1B and 2B: Comparison

<table>
<thead>
<tr>
<th>1B</th>
<th>2B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Łukasiewicz tribe</td>
<td>product or strict Frank tribe</td>
</tr>
<tr>
<td>product as an additional operation</td>
<td>Łukasiewicz operations derived</td>
</tr>
<tr>
<td>(M2) $A \odot B = 0 \Rightarrow$</td>
<td>(M2*) $P(A \oplus B)$</td>
</tr>
<tr>
<td></td>
<td>$= P(A) + P(B) - P(A \odot B)$</td>
</tr>
<tr>
<td>(M4) satisfied</td>
<td>(M4) required</td>
</tr>
<tr>
<td></td>
<td>$A_n \downarrow A \Rightarrow P(A_n) \rightarrow P(A)$</td>
</tr>
</tbody>
</table>

The same structure and definition of probability measure
(in agreement with the original Zadeh’s suggestion,
now correctly defined using axiomatics similar to the classical theory).
1B and 2B: Comparison

<table>
<thead>
<tr>
<th>1B</th>
<th>2B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Łukasiewicz tribe</td>
<td>product or strict Frank tribe</td>
</tr>
<tr>
<td>product as an additional operation</td>
<td>Łukasiewicz operations derived</td>
</tr>
<tr>
<td>(M2) $A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B)$</td>
<td>(M2*) $P(A \oplus B) = P(A) + P(B) - P(A \odot B)$</td>
</tr>
<tr>
<td>(M4) satisfied</td>
<td>(M4) required</td>
</tr>
<tr>
<td>$A_n \searrow A \Rightarrow P(A_n) \rightarrow P(A)$</td>
<td></td>
</tr>
</tbody>
</table>

The same structure and definition of probability measure (in agreement with the original Zadeh’s suggestion, now correctly defined using axiomatics similar to the classical theory).

Only (nearly) Frank t-norms are convenient (as predicted by Butnariu and Klement), others do not admit $\sigma$-order continuous probability measures (and other measures lack motivation).
Conditional probability (of classical events)

Probability measure \( P(\cdot) \) is considered as a \textbf{mixture} of probabilities \( P(\cdot|B) \) (for the case when \( B \) occurs), \( P(\cdot|B') \) (for the case when \( B \) does not occur).
Conditional probability (of classical events)

Probability measure $P(.)$ is considered as a mixture of probabilities $P(.|B)$ (for the case when $B$ occurs), $P(.|B')$ (for the case when $B$ does not occur).

We obtain the formula for total probability

$$P(A) = P(B) P(A|B) + P(B') P(A|B').$$
Conditional probability (of classical events)

Probability measure $P(.)$ is considered as a mixture of probabilities $P(.|B)$ (for the case when $B$ occurs), $P(.|B')$ (for the case when $B$ does not occur).

We obtain the formula for total probability

$$P(A) = P(B) P(A|B) + P(B') P(A|B').$$

$P(.|B), P(.|B')$ are determined by $P(.)$ and $P(B|B') = 0, P(B'|B) = 0$, having a unique solution for $A = (A \odot B) \oplus (A \odot B')$,

$$P(A) = P(A \odot B) + P(A \odot B'),$$

$$P(B) P(A|B) = P(A \odot B), \quad P(B') P(A|B') = P(A \odot B')$$

(unless $P(B)$ or $P(B')$ is zero; then the respective conditional probability is not determined).
Conditional probability (of fuzzy events)

The classical approach works if $B \in \mathcal{B}(T)$; then $A = (A \odot B) \oplus (A \odot B')$, ...
Conditional probability (of fuzzy events)

The classical approach works if $B \in \mathcal{B}(T)$; then $A = (A \odot B) \oplus (A \odot B')$, ...

$\Rightarrow$ we may condition by Boolean events.
Conditional probability (of fuzzy events)

The classical approach works if $B \in \mathcal{B}(T)$; then $A = (A \circ B) \oplus (A \circ B')$, ...

$\Rightarrow$ we may condition by Boolean events.

Mixtures are always well defined.
Conditional probability (of fuzzy events)

The classical approach works if $B \in \mathcal{B}(T)$; then $A = (A \odot B) \oplus (A \odot B')$, ...

$\Rightarrow$ we may condition by Boolean events.

Mixtures are always well defined.

**Problem** [Riečan, Mundici]: Conditioning by a fuzzy event, e.g., $H = 1/2 = H'$. 
Conditional probability (of fuzzy events)

The classical approach works if $B \in \mathcal{B}(T)$; then $A = (A \circ B) \oplus (A \circ B')$, ...

⇒ we may condition by Boolean events.

Mixtures are always well defined.

**Problem** [Riečan, Mundici]: Conditioning by a fuzzy event, e.g., $H = 1/2 = H'$.

We cannot even have $P(H|H) = 1$. 
Conditional probability (of fuzzy events)

The classical approach works if $B \in \mathcal{B}(T)$; then $A = (A \odot B) \oplus (A \odot B')$, ...

$\Rightarrow$ we may condition by Boolean events.

Mixtures are always well defined.

**Problem** [Riečan, Mundici]: Conditioning by a fuzzy event, e.g., $H = 1/2 = H'$.

We cannot even have $P(H|H) = 1$.

“I said neither YES nor NO, but I WAS COMPLETELY RIGHT!”
Conditional probability (of fuzzy events)

The classical approach works if $B \in \mathcal{B}(T)$; then $A = (A \odot B) \oplus (A \odot B')$, ...

$\Rightarrow$ we may condition by Boolean events.

Mixtures are always well defined.

**Problem** [Riečan, Mundici]: Conditioning by a fuzzy event, e.g., $H = 1/2 = H'$.

We cannot even have $P(H|H) = 1$.

“I said neither YES nor NO, but I WAS COMPLETELY RIGHT!”

In reasonable models, $P(H) = 1/2$. 
Conditional probability (of **fuzzy** events)

The classical approach works if $B \in \mathcal{B}(T)$; then $A = (A \odot B) \oplus (A \odot B')$, ...

$\Rightarrow$ we may condition by Boolean events.

Mixtures are always well defined.

**Problem** [Riečan, Mundici]: Conditioning by a fuzzy event, e.g., $H = 1/2 = H'$.

We cannot even have $P(H|H) = 1$.

"I said neither YES nor NO, but I WAS COMPLETELY RIGHT!"

In reasonable models, $P(H) = 1/2$.

If $H$ has been observed, it gives no information.
Conditional probability (of fuzzy events)

The classical approach works if $B \in \mathcal{B}(T)$; then $A = (A \odot B) \oplus (A \odot B')$, ...

$\Rightarrow$ we may condition by Boolean events.

Mixtures are always well defined.

**Problem** [Riečan, Mundici]: Conditioning by a fuzzy event, e.g., $H = 1/2 = H'$.

We cannot even have $P(H|H) = 1$.

"I said neither YES nor NO, but I WAS COMPLETELY RIGHT!"

In reasonable models, $P(H) = 1/2$.

If $H$ has been observed, it gives no information.

$\Rightarrow P(A|H) = P(A)$ [Kroupa]
Conditional probability (of fuzzy events)

The classical approach works if $B \in \mathcal{B}(T)$; then $A = (A \odot B) \oplus (A \odot B')$, ...

$\Rightarrow$ we may condition by Boolean events.

Mixtures are always well defined.

**Problem** [Riečan, Mundici]: Conditioning by a fuzzy event, e.g., $H = 1/2 = H'$.

We cannot even have $P(H|H) = 1$.

“I said neither YES nor NO, but I WAS COMPLETELY RIGHT!”

In reasonable models, $P(H) = 1/2$.

If $H$ has been observed, it gives no information.

$\Rightarrow P(A|H) = P(A)$ [Kroupa]

This is not only a question of formulas, but also interpretation.