Fuzzy control

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Outline:

- Historical introduction
- Brief overview of classical control theory
- Principles of fuzzy control

Basics of classical control theory

Intuitively used in ancient times.

Watt – steam engine
negative feedback:
high speed $\Rightarrow$ less steam
low speed $\Rightarrow$ more steam

Watt did not care of non-linearity and dynamical properties of the controller.

It sufficed to have it very sensitive and much faster than the controlled system.

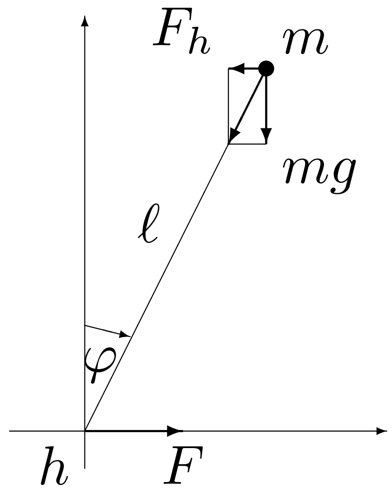
Middle of 20th century: [Wiener, Shannon, Nyquist, ... , Zadeh]...

Inspiration: Cartpole problem (inverted pendulum)

Very simplified model:

- no friction ($\Rightarrow$ no damping),
- zero moment of inertia (**single pendulum**); a real pendulum is described by the same model with some **equivalent length**,
- the influence of the mass of the pendulum on our movements is neglected,
- **linearization** around $\varphi = 0$,
- the acceleration (and the fluctuations of forces) in the vertical direction are neglected.

Inspiration: Cartpole problem (inverted pendulum)



Constants:
$\ell =$ lenght of the pendulum
$m =$ mass of the pendulum
$g =$ acceleration of gravity

Variables:
$t =$ time
$h(t) =$ horizontal coordinate of the axis of the pendulum
$\varphi(t) =$ angle of the pendulum (measured from the vertical direction)
$a(t) =$ acceleration of the axis of the pendulum in the horizontal direction
(proportional to the acting force $F(t)$)
$q(t) = h(t) + \ell \sin \varphi(t) =$ horizontal coordinate of the mass of the pendulum

Inspiration: Cartpole problem (inverted pendulum)

Equations (with (t) omitted):

$$q = h + \ell \sin\varphi\,.$$

Linearized coordinate of the mass of the pendulum:

$$q = h + \ell\,\varphi\,,$$

Its second derivative is proportional to the horizontal force on the pendulum:

$$\frac{F_h}{m} = \ddot{h} + \ell\,\ddot{\varphi}\,.$$

The pendulum transfers only a force parallel to it, hence

$$\frac{F_h}{mg} = \tan\varphi\,, \qquad \text{linearized:} \qquad \frac{F_h}{m} = g\varphi\,.$$

We act through a force causing an acceleration

$$a = \ddot{h}\,.$$

The dynamics of the system is described by a system of linear ODEs:

$$\begin{aligned} \ddot{h} + \ell\,\ddot{\varphi} &= g\,\varphi\,,\\ \ddot{h} &= a\,. \end{aligned}$$

Inspiration: Cartpole problem (inverted pendulum)

State variables:

$$x_1 = \varphi\,, \qquad x_2 = \dot{\varphi}\,, \qquad x_3 = h\,, \qquad x_4 = \dot{h}\,, \qquad \mathbf{x} = \begin{bmatrix} \varphi \\ \dot{\varphi} \\ h \\ \dot{h} \end{bmatrix}.$$

Control variable (input):

$$u_1 = a\,, \qquad \mathbf{u} = \begin{bmatrix} a \end{bmatrix}.$$

The linearized dynamics of the system is described by

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\,, \\
\dot{x}_1 = \dot{\varphi} &= x_2\,, \\
\dot{x}_2 = \ddot{\varphi} &= \frac{g}{\ell}\varphi - \frac{1}{\ell}a = \frac{g}{\ell}x_1 - \frac{1}{\ell}u_1\,, \\
\dot{x}_3 = \dot{h} &= x_4\,, \\
\dot{x}_4 = \ddot{h} &= a = u_1\,.
\end{aligned}$$

Inspiration: Cartpole problem (inverted pendulum)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g}{\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{-1}{\ell} \\ 0 \\ 1 \end{bmatrix}.$$

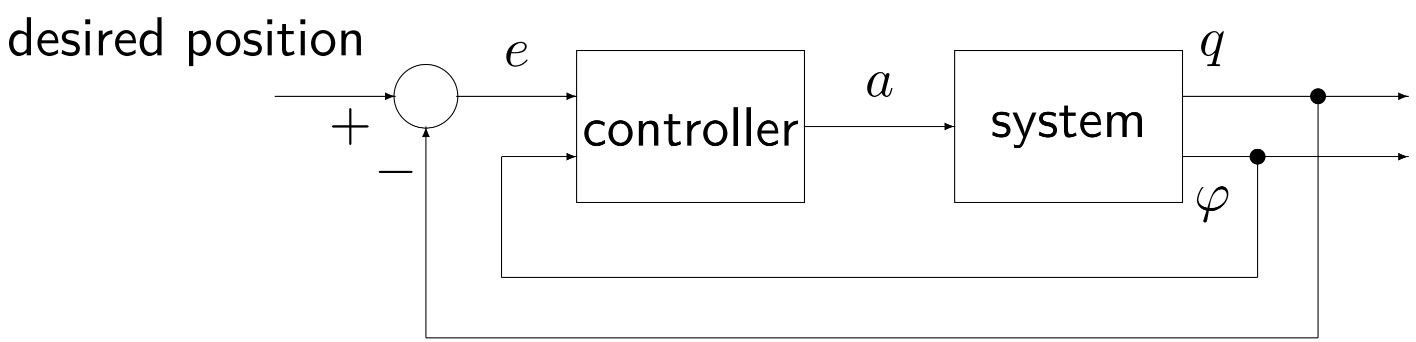
Inspiration: Cartpole problem (inverted pendulum)

Output variables:

$$y_1 = \varphi\,, \qquad y_2 = q = h + \ell\varphi\,, \qquad \mathbf{y} = \begin{bmatrix} \varphi \\ q \end{bmatrix}, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\ .$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \ell & 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

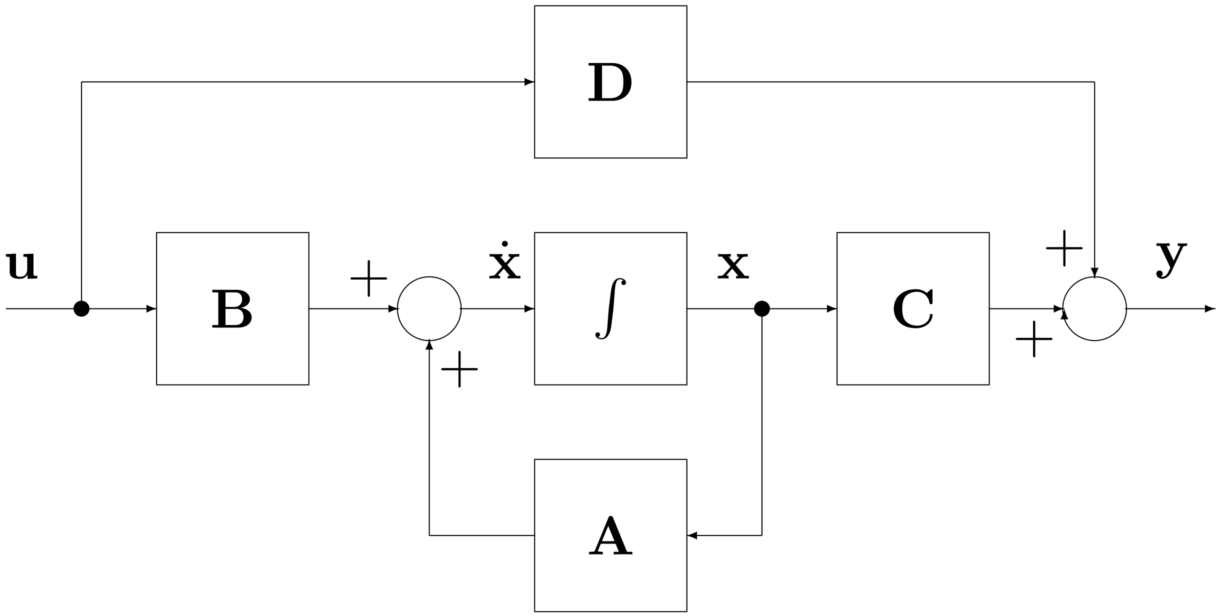
(no direct influence of the control on the output).



Descriptions of a linear dynamical system

Internal description: $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \ ,$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \ .$$



External description: Using the Laplace images $\mathbf{U}(s), \mathbf{Y}(s)$ of $\mathbf{u}(t), \mathbf{y}(t)$ (with zero initial conditions),

$\mathbf{Y}(s) = \mathbf{G}(s)\,\mathbf{U}(s)$, where $\mathbf{G}(s)$ is the **transfer function** of the system.

Having vectors of variables, $\mathbf{G}(s)$ is a matrix of functions of s.

Its element $\mathbf{G}_{ij}(s)$ is the Laplace image of the response at the jth output to the Dirac pulse at the ith input (difficult to measure directly).

How to derive the external description from the internal one?

Under zero initial conditions ($\mathbf{I}$ denotes the unit matrix):

$$\begin{aligned} s\,\mathbf{X}(s) &= \mathbf{A}\,\mathbf{X}(s) + \mathbf{B}\,\mathbf{U}(s)\,, \\ \mathbf{Y}(s) &= \mathbf{C}\,\mathbf{X}(s) + \mathbf{D}\,\mathbf{U}(s)\,, \end{aligned}$$

$$\begin{aligned} \mathbf{X}(s) &= (s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{U}(s)\,, \\ \mathbf{Y}(s) &= \underbrace{\left(\mathbf{C}\,(s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}\right)}_{\mathbf{G}(s)}\,\mathbf{U}(s)\,. \end{aligned}$$

$\det(s\,\mathbf{I} - \mathbf{A}) = 0$ is the **characteristic equation** of the system.
Its solutions (in variable s) are **eigenvalues**.
Each of them corresponds to one exponential component of the solution.
The system is stable iff all characteristic numbers have negative real parts
(this can be tested without finding the characteristic numbers, which is a difficult problem).

How to derive the internal description from the external one?
This is not unique, there are many standardized methods whose choice depends on the hardware implementation.

Non-stability of the cartpole

$$s\,\mathbf{I}-\mathbf{A}=\begin{bmatrix} s & -1 & 0 & 0 \\ -\frac{g}{\ell} & s & 0 & 0 \\ 0 & 0 & s & -1 \\ 0 & 0 & 0 & s \end{bmatrix},$$

$$\det(s\,\mathbf{I}-\mathbf{A})=\frac{1}{\ell}\left(s^2\ell-g\right)s^2\,.$$

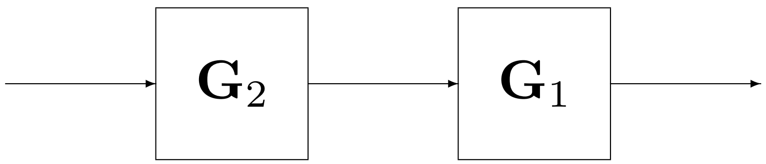
This polynomial has roots 0 (double) and $\pm\sqrt{\frac{g}{\ell}}$.
All of them are real, one positive (causing non-stability), one zero (at the boundary of stability region) and one negative (corresponding to a stable component).

$$(s\,\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix} \frac{-\ell s}{g-s^2\ell} & \frac{-\ell}{g-s^2\ell} & 0 & 0 \\ \frac{-g}{g-s^2\ell} & \frac{-\ell s}{g-s^2\ell} & 0 & 0 \\ 0 & 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & 0 & \frac{1}{s} \end{bmatrix},$$

$$\mathbf{G}(s)=\mathbf{C}\,(s\,\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{B}+\mathbf{D}=\begin{bmatrix} \frac{1}{g-s^2\ell} \\ \frac{1}{s^2}+\frac{\ell}{g-s^2\ell} \end{bmatrix}.$$

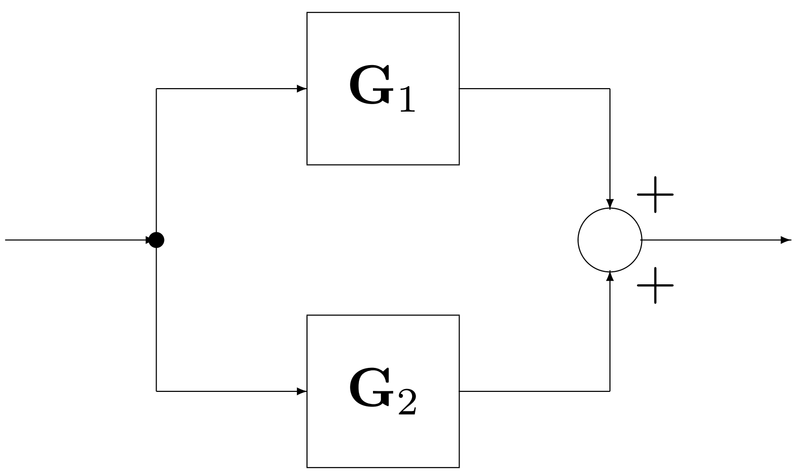
Connections of dynamical systems

Series connection:



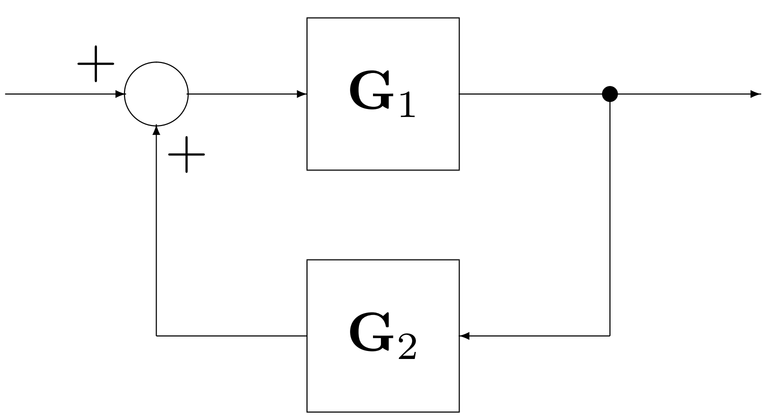
$$\mathbf{G}(s) = \mathbf{G}_1(s)\,\mathbf{G}_2(s)$$

Parallel connection:



$$\mathbf{G}(s) = \mathbf{G}_1(s) + \mathbf{G}_2(s)$$

Feedback connection:



$$\mathbf{G}(s) = \mathbf{G}_1(s)\left(\mathbf{I} - \mathbf{G}_2(s)\,\mathbf{G}_1(s)\right)^{-1}$$

Exceptions: Cancellation of roots

Example of a series connection:

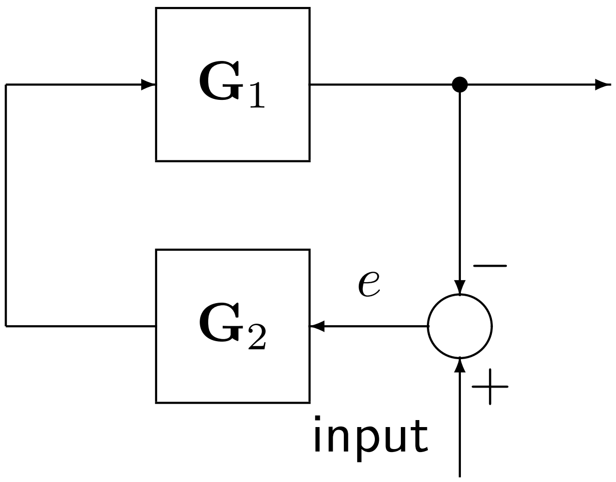
$$\begin{aligned}
\mathbf{G}(s) &= \mathbf{G}_1(s)\,\mathbf{G}_2(s)\,, \\
\mathbf{G}_1(s) &= \frac{1}{s+a}\,, \\
\mathbf{G}_2(s) &= \frac{s+a}{s+b}\,, \\
\mathbf{G}(s) &= \frac{1}{s+b}\,.
\end{aligned}$$

The factor $s+a$ has been cancelled and does not occur in the external description, although it corresponds to some internal part.
Moreover, for $a<0$ it is unstable, while the stability of the external description depends only on b.

This part has a state variable which is not controllable, neither observable.

Classical controller design

We usually put the controller in the feedback loop:



$$\mathbf{G}(s) = \mathbf{G}_1(s)\,\mathbf{G}_2(s)\,\left(\mathbf{I} + \mathbf{G}_2(s)\,\mathbf{G}_1(s)\right)^{-1}$$

The stability of the whole loop is influenced by the factor $\left(\mathbf{I} + \mathbf{G}_2(s)\,\mathbf{G}_1(s)\right)^{-1}$, where $\mathbf{G}_1(s)$ (the controlled system) is given and $\mathbf{G}_2(s)$ (the controller) can be chosen almost arbitrarily.

Using the above analysis, we can decide the stability of the loop with the proposed controller (difficult).

The task is easier if we have more information than the output of the controlled system, in particular if we can measure the **states**; then a feedback from states allows – in its extreme (theoretical) form – to achieve arbitrary dynamics of the control loop.

Generally, the more information we have the better the control behaviour can be.

Structure of classical controllers

Usually the controller uses a linear combination of its inputs (this is **P**roportional to the signal), its **I**ntegrals, and **D**erivatives (**PID controller**).

Remark:
In fact, a derivative cannot be correctly computed in real time (it requires information which is not available); even the causality principle and boundedness of power prove the impossibility of an element performing the derivative; only a rough approximation is used instead.

Higher order integrals are possible, but not used so often.

The controller is usually implemented as another (inner) feedback system with an amplifier in the direct branch and a suitable feedback.
In the one-dimensional case, for a constant $\mathbf{G}_1(s) \to \infty$ the feedback results in

$$\mathbf{G}(s) = \frac{\mathbf{G}_1(s)}{1 + \mathbf{G}_2(s)\,\mathbf{G}_1(s)} = \frac{1}{\frac{1}{\mathbf{G}_1(s)} + \mathbf{G}_2(s)} \to \frac{1}{\mathbf{G}_2(s)}$$

and the feedback loop “fully" determines the properties.

Problems of the classical control

- Non-linearity.
- Boundedness of control variables.
- Further requirements on control (e.g., zero overshoot) which cannot be easily checked in the model.
- Parameters are not precisely known (or it is difficult to measure them).
- Sensitivity to changes of parameters and input values.
- Discretization.
- Delays in actions (e.g., computation of the control variable).
- Non-stationarity (the parameters change).
- The model does not describe all important relations (it is drastically simplified).
- Problems of solvability of the task.
- Non-interpretability of the parameters of the controller.

Brief history of fuzzy control

[Zadeh 1973] suggested the use of fuzzy logic in control (he already contributed to the development of the classical control theory).

[Mamdani, Assilian 1975]: the first fuzzy controller (of a steam engine).

[Holmblad 1982]: a fuzzy controller of a cement kiln (high non-linearity, many variables, manual control used before).

[Sugeno 1985]: prototypes of other industrial applications.

[Yasunubo et al. 1983]: a fuzzy controller of the Sendai Underground (operating since 1987).

A boom of fuzzy controllers in 80's and 90's, mainly in Japan (now mainly washing machines, vacuum cleaners, camcorders, etc.).

Now it is time to test whether fuzzy control may infiltrate in more difficult and demanding applications.

Basic ideas and notions of fuzzy control

Inputs of a fuzzy controller:

- desired output values,
- actual output values,
- possibly internal states of the controlled system,
- event. additional information from the user, usually linguistic.

Input variables are coordinates in the **input space**, $\mathcal{X}$, usually a convex subset of R^μ.

Outputs of a fuzzy controller:

- control actions (inputs of the controlled system),
- event. additional information for the user.

Output variables are coordinates in the **output space**, $\mathcal{Y}$, usually a convex subset of R^ν.

Crisp controller

A classical controller performs a mapping $f\colon \mathcal{X} \to \mathcal{Y}$.
It can be represented by a crisp subset of $\mathcal{X} \times \mathcal{Y}$, namely

$$\{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid y = f(x)\},$$

and by a (crisp) membership function $R\colon \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$,

$$R(x, y) = \begin{cases} 1 & \text{if } y = f(x), \\ 0 & \text{otherwise.} \end{cases}$$

Motivation of a fuzzy controller

Sometimes a human expert (or other source, e.g., data mining) can give us hints in the form

if input is ... **then** output is ... **and**

...

if $x \in A_n$ **then** $y \in C_n$,

(rule base of **if-then rules**).

The rules are vague, often with unsharp boundaries of applicability.

Fuzzy controller

Zadeh's suggestion (1973): Express the information from the rule base using fuzzy sets, as a fuzzy relation $R\colon \mathcal{X} \times \mathcal{Y} \to [0,1]$
(a fuzzy subset of $\mathcal{X} \times \mathcal{Y}$, $R \in \mathcal{F}(\mathcal{X} \times \mathcal{Y})$) ,
which generalizes the classical control function.

Moreover, the internal **inference mechanism** can work with fuzzy subsets of the input/output space (instead of points) and map fuzzy subsets of the input space $\mathcal{X}$ onto fuzzy subsets of the output space $\mathcal{Y}$,

$$\Phi\colon \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{Y})\,.$$

The input can be fuzzy, but it is often crisp; sometimes a crisp input is fuzzified.

The output can be fuzzy, but usually a crisp value is required; a **defuzzification** $\Delta\colon \mathcal{F}(\mathcal{Y}) \to \mathcal{Y}$ takes place as the final step.

For fuzzy inference itself, we need a correspondence between **fuzzy** subsets of the input and output spaces.

Basic notions of a fuzzy controller

Rule database:

if X is A_1 **then** Y is C_1 **and**
$\dots$
if X is A_n **then** Y is C_n ,

where
$X \in \mathcal{F}(\mathcal{X})$ is a fuzzy input,
$Y = \Phi(X) \in \mathcal{F}(\mathcal{Y})$ is the corresponding fuzzy output,
$A_i \in \mathcal{F}(\mathcal{X})$, $i = 1, \dots, n$, are **antecedents** (**premises**) which can be interpreted as

- assumptions,
- domains of applicability, or
- typical fuzzy inputs.

$C_i \in \mathcal{F}(\mathcal{Y})$, $i = 1, \dots, n$, are **consequents** (**conclusions**) expressing the desired outputs.

Dimensionality

Antecedents are subsets of multi-dimensional spaces.
They carry information about several variables (and so do the consequents).
Usually they are decomposed to conjunctions (cylindric extensions) of one-dimensional fuzzy sets.
Then the rules attain the form

if A_1 is A_{i1}
and ...
and A_μ is $A_{i\mu}$
then C_1 is C_{i1}
and ...
and C_ν is $C_{i\nu}$;

$i = 1, \ldots, n\,.$

If an antecedent has a more complex shape (non-convex), we may cover it approximately by several rules of the above form.

Simplifying assumptions

1. We ignore the conjunctions (cylindric extensions) and admit arbitrary shapes of antecedents.

2. We decompose the output to single variables considered independently. Without loss of generality, we restrict attention to MISO (Mulitple Input Single Output) systems.

Compositional rule of inference

The rule base is represented by a fuzzy relation $R \in \mathcal{F}(\mathcal{X} \times \mathcal{Y})$.
The output, Y, is obtained by a composition of R with the input, X:

$$Y = \Phi(X) = X \underset{\cdot}{\circ} R\,,$$

$$Y(y) = \sup_{x \in \mathcal{X}} \big(R(x,y) \underset{\cdot}{\wedge} X(x) \big)\,,$$

where $\underset{\cdot}{\wedge}$ is a t-norm (fuzzy conjunction); different choices are possible, but we shall restrict to **continuous t-norms**.

The supremum is the standard t-conorm; it should not be replaced by another t-conorm (because it may have uncountably many arguments).

How to derive the fuzzy relation R from the rule base

The most natural idea: **Residuum-based fuzzy controller**:

$$R_{\mathsf{RES}}(x,y) = \min_i \big(A_i(x) \underset{\cdot}{\rightarrow} C_i(y)\big)\,,$$

where $\underset{\cdot}{\rightarrow}$ is a **fuzzy implication**, usually the **residuum** (**R-implication**) of $\underset{\cdot}{\wedge}$,

$$\alpha \underset{\cdot}{\rightarrow} \beta = \sup\{\gamma \in [0,1] \mid \gamma \underset{\cdot}{\wedge} \alpha \le \beta\}\,.$$

Properties of residua of continuous t-norms:

- $\alpha \underset{\cdot}{\rightarrow} \beta = 1$ iff $\alpha \le \beta$,
- $1 \underset{\cdot}{\rightarrow} \beta = \beta$,
- non-increasing in the first and non-decreasing in the second variable,
- continuous iff the t-norm $\underset{\cdot}{\wedge}$ is nilpotent,
- **adjointness**: $a \underset{\cdot}{\wedge} b \le c$ iff $a \le b \underset{\cdot}{\rightarrow} c$.

How to derive the fuzzy relation R from the rule base 2

Mamdani–Assilian fuzzy controller:

$$R_{\mathsf{MA}}(x,y) = \max_i \bigl(A_i(x) \mathbin{\dot\wedge} C_i(y)\bigr)\,.$$

Logically this is a **disjunction of conjunctions**, not a conjunction of implications.

These expressions are not totally different; if **crisp** sets A_i, $i = 1, \ldots, n$, form a partition of $\mathcal{X}$ (i.e., they are mutually disjoint and $\bigcup_i A_i = \mathcal{X}$), then $R_{\mathsf{MA}} = R_{\mathsf{RES}}$.

However, this usually is not the case.

Comparison of residuum-based and Mamdani–Assilian controllers

Continuity:

R_{RES} only for $\underset{\cdot}{\wedge}$ nilpotent,
R_{MA} always.

Computational efficiency:

$$\Phi_{\mathsf{RES}}(X)(y) = \sup_x \Big(X(x) \underset{\cdot}{\wedge} \min_i (A_i(x) \underset{\cdot}{\to} C_i(y)) \Big)$$

requires **three** nested cycles (over $\mathcal{X}$ and $\mathcal{Y}$ and over the number of rules).

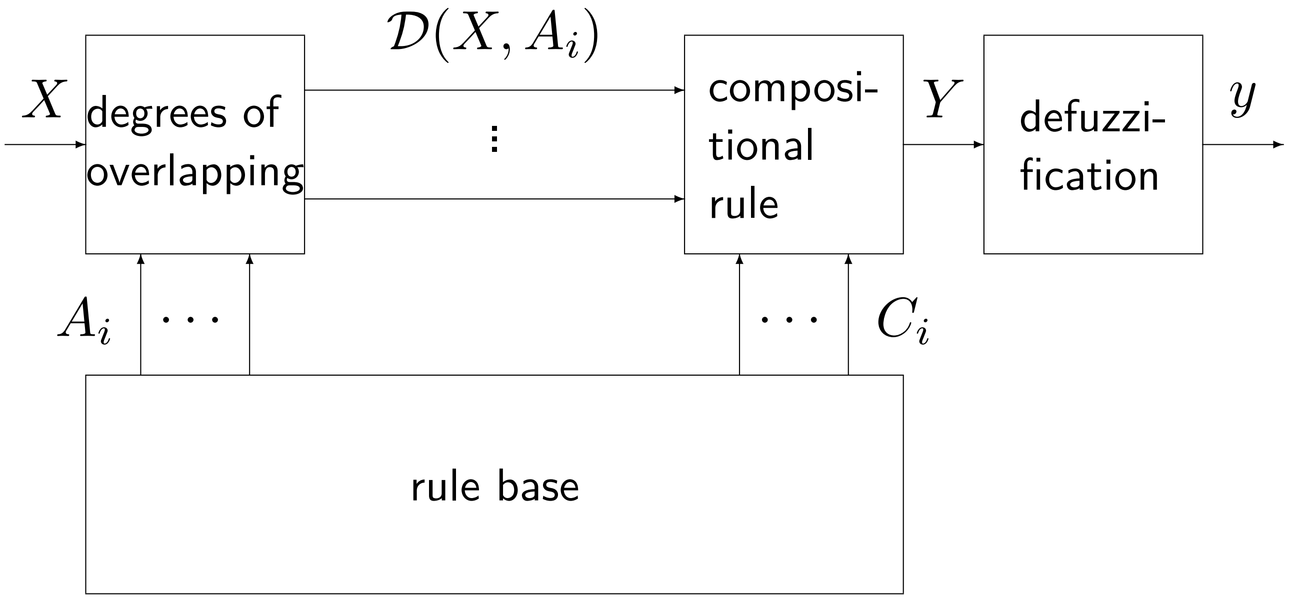
$$\begin{aligned} \Phi_{\mathsf{MA}}(X)(y) &= \sup_x \Big(X(x) \underset{\cdot}{\wedge} \max_i (A_i(x) \underset{\cdot}{\wedge} C_i(y)) \Big) \\ &= \max_i \sup_x \Big(X(x) \underset{\cdot}{\wedge} A_i(x) \underset{\cdot}{\wedge} C_i(y) \Big) \\ &= \max_i (\mathcal{D}(X, A_i) \underset{\cdot}{\wedge} C_i(y)) \,. \end{aligned}$$

$\mathcal{D}(X, A_i) = \sup_x (X(x) \underset{\cdot}{\wedge} A_i(x))$ ... the **degree of overlapping (non-disjointness)**, here equal to the **degree of firing (applicability)**.

Tequires **two** nested cycles (over $\mathcal{X}$ and the number of rules) resulting in real numbers $\mathcal{D}(X, A_i)$, $i = 1, \ldots, n$; **then two** nested cycles (over $\mathcal{Y}$ and the number of rules).

Φ_{MA} can be computed more efficiently (approx. $\#Y/2$-times faster).

Principle of Mamdani–Assilian controller — block diagram



Generalized Compositional Rule of Inference

[Thiele 1995], [Lehmke, Reusch, Temme, Thiele 1998]
FATI principle (First Aggregation, Then Inference)

$$R_{\mathsf{FATI}}(x,y) = \beta\big(\pi_1\big(A_1(x),C_1(y)\big),\dots,\pi_n\big(A_n(x),C_n(y)\big)\big)\,,$$

where $\pi_i\colon [0,1]^2 \to [0,1], \quad \beta\colon [0,1]^n \to [0,1]\,.$

$$Y(y) = \Phi_{\mathsf{FATI}}(X)(y) = Q\{\kappa\big(X(x),R_{\mathsf{FATI}}(x,y)\big) \mid x \in \mathcal{X}\}\,,$$

where $\kappa\colon [0,1]^2 \to [0,1], \quad Q\colon \mathcal{P}([0,1]) \to [0,1]$ (almost arbitrary operations).

Particular cases:
Mamdani–Assilian controller: $\pi_i = \dot{\wedge}, \quad \beta = \max, \quad \kappa = \dot{\wedge}, \quad Q = \sup.$

Residuum-based controller: $\pi_i = \dot{\to}, \quad \beta = \min, \quad \kappa = \dot{\wedge}, \quad Q = \sup.$

Generalized Compositional Rule of Inference 2

[Thiele 1995], [Lehmke, Reusch, Temme, Thiele 1998]
FITA principle (First Inference, Then Aggregation)

$$R_{\mathsf{FITA}i}(x,y) = \pi_i\big(A_i(x), C_i(y)\big)\,,$$

where $\pi_i\colon [0,1]^2 \to [0,1]\,.$

$$\Phi_{\mathsf{FITA}i}(X)(y) = Q_i\big\{\kappa_i\big(X(x), R_{\mathsf{FITA}i}(x,y)\big) \mid x \in \mathcal{X}\big\}\,,$$

where $\kappa_i\colon [0,1]^2 \to [0,1],\quad Q_i\colon \mathcal{P}([0,1]) \to [0,1]\,.$

$$Y(y) = \Phi_{\mathsf{FITA}}(X)(y) = \alpha\big(\Phi_{\mathsf{FITA}1}(X)(y), \ldots, \Phi_{\mathsf{FITA}n}(X)(y)\big)\,,$$

where $\alpha\colon [0,1]^n \to [0,1].$

Particular cases:
FITA Mamdani–Assilian controller: $\pi_i = \dot\wedge,\quad \kappa_i = \dot\wedge,\quad Q_i = \sup,\quad \alpha = \max.$

FITA Residuum-based controller: $\pi_i = \dot\to,\quad \kappa_i = \dot\wedge,\quad Q_i = \sup,\quad \alpha = \min.$

Requirements on the rule base [Driankov et al. 1993]

1. **Completeness**: $\bigcup_i \mathrm{Supp}\, A_i = \mathcal{X}$, where Supp $A_i = \{x \in \mathcal{X} \mid A_i(x) > 0\}$.
2. **Consistency**: $(A_i = A_j) \Rightarrow (C_i = C_j)$ (rather weak condition, otherwise, there is **no model** of such a rule base).
3. **Continuity**: $(A_i, A_j$ are "neighbouring antecedents" $) \Rightarrow (C_i \cap C_j \neq \emptyset)$.
   usually
   $$(A_i \cap A_j \neq \emptyset) \Rightarrow (C_i \cap C_j \neq \emptyset)\,.$$
4. **Interaction** or **(local) correctness** ([Thiele 1995]): $\forall j : \Phi(A_j) = C_j$.

   The output of the controller should be the fuzzy union of the outputs of separate rules (i.e., FATI=FITA);
   this weaker form always holds for a Mamdani–Assilian controller;
   see also [Lehmke, Reusch, Temme, Thiele 1998] for more general sufficient conditions for this equality).

Recommendations on the rule base [Driankov et al. 1993]

Antecedents (one-dimensional) should be

- **normal**, $\forall i \; \exists x \in \mathcal{X} : A_i(x) = 1$,
- continuous,
- symmetric (when possible, usually not at the borders of the input space!).

The recommended degree of overlapping of neighbouring antecedents (computed using the standard t-norm, $\min$) is 0.5.

The recommended endpoints of the support of an antecedent are the peeks of the neighbouring antecedents.

Requirements on the rule base [Moser, Navara 2002]

- **Local correctness** (**interaction**): $\forall j : \Phi(A_j) = C_j$.
- **Strong completeness**: $\forall$ normal $X \in \mathcal{F}(\mathcal{X}) : \Phi(X) \not\subseteq \bigcap_i C_i$, where the fuzzy intersection is standard (computed using $\min$).
- **Weak interpolation property**: $\Phi(X)$ is in the convex hull of all C_i with i such that $\operatorname{Supp} A_i \cap \operatorname{Supp} X \neq \emptyset$.
- **Crisp correctness** (**crisp interaction**): $\big(A_i(x) = 1\big) \Rightarrow \big(\Phi(x) = \Phi(\{x\}) = C_i\big)$ ("if there is a totally firing rule, it determines the output").

Completeness of the rule base

Completeness is required, because in any situation we need at least one firing rule.

Nevertheless, non-completeness is sometimes tolerated for the following reasons:

- In expert systems; "I don't know" could be a legitimate answer (of an expert system, not of a pilot!).
- The input is impossible (then do not include it in the input space!).
- The input values are fuzzified so that they always overlap with some antecedent.
- The sparse database is used for interpolation [Kóczy et al. 1997].
- Some inputs do not require any action (we just wait until the situation changes).

The latter case can be formally described by an additional "**else rule**"
[Amato, Di Nola, Navara 2003].
It is treated differently w.r.t. other requirements.
In any case, it assumes that we assign a meaning of "no action".
the output variable has to be defined always.

Completeness of the rule base

Omitting rules for some situations is motivated by the attempt to reduce the number of rules (**curse of dimensionality**).

Sometimes the table of linguistic rules does not cover some combinations of linguistic variables.
This does not obviously mean that the antecedents are not complete; the case may be covered by neighbouring rules, although with a smaller degree of firing.

Correctness of Mamdani–Assilian controller

When $\forall j : \Phi(A_j) = A_j \underset{\cdot}{\circ} R_{\mathsf{MA}} = C_j$? (A system of fuzzy relational equations for a fuzzy relation R_{MA}.)

For Mamdani–Assilian controller:

Theorem: $\forall j : \Phi_{\mathsf{MA}}(A_j) \geq C_j$.

Proof: $X := A_j$,
$\mathcal{D}(X, A_j) = \mathcal{D}(A_j, A_j) = 1$ (due to normality),
$\Phi_{\mathsf{MA}}(A_j)(y) = \max_i (\mathcal{D}(A_j, A_i) \underset{\cdot}{\wedge} C_i(y)) \geq \underbrace{\mathcal{D}(A_j, A_j)}_{1} \underset{\cdot}{\wedge} C_j(y) = C_j(y)$.

Correctness of Mamdani–Assilian controller

Theorem [de Baets 1996, Perfilieva, Tonis 1997]: $\big(\forall j : \Phi_{\mathsf{MA}}(A_j) = C_j\big)$ iff $\big(\forall i\ \forall j : \mathcal{D}(A_i, A_j) \le \mathcal{I}(C_i, C_j)\big)$,
where $\mathcal{I}(C_i, C_j) = \inf_y \big(C_i(y) \dot{\to} C_j(y)\big)$
(the implication $\dot{\to}$ has to be the residuum of $\dot{\wedge}$).

Instead of $I(C_i, C_j)$ we may use $\mathcal{E}(C_i, C_j) = \inf_y \big(C_i(y) \dot{\leftrightarrow} C_j(y)\big)$
(**degree of indistinguishability (equality)**),
where $\alpha \dot{\leftrightarrow} \beta = \min(\alpha \dot{\to} \beta, \beta \dot{\to} \alpha) = (\alpha \dot{\to} \beta) \dot{\wedge} (\beta \dot{\to} \alpha)$.

Proof: The negation of the left-hand side is

$$\begin{aligned}
\exists j\ \exists y : \Phi_{\mathsf{MA}}(A_j)(y) &> C_j(y)\,,\\
\exists j\ \exists y\ \exists x : A_j(x) \dot{\wedge} R_{\mathsf{MA}}(x, y) &> C_j(y)\,,\\
\exists i\ \exists j\ \exists y\ \exists x : A_j(x) \dot{\wedge} A_i(x) \dot{\wedge} C_i(y) &> C_j(y)\,,\\
\exists i\ \exists j\ \exists y\ \exists x : A_j(x) \dot{\wedge} A_i(x) &> C_i(y) \dot{\to} C_j(y)\,,\\
\exists i\ \exists j : \sup_x \big(A_j(x) \dot{\wedge} A_i(x)\big) &> \inf_y \big(C_i(y) \dot{\to} C_j(y)\big)\,,
\end{aligned}$$

which is the negation of the right-hand side.

Correctness of Mamdani–Assilian controller [Moser, Navara 1999]

If $\underset{\cdot}{\wedge}$ has no zero divisors (e.g., the minimum or product), then $\mathcal{D}(A_i, A_j) \leq \mathcal{E}(C_i, C_j)$ is satisfied in two situations:

- $\mathcal{E}(C_i, C_j) > 0$; then $\operatorname{Supp} C_i = \operatorname{Supp} C_j$, which is rather unusual,
- $\mathcal{E}(C_i, C_j) = 0$; then $\mathcal{D}(A_i, A_j) = 0$, $\operatorname{Supp} A_i \cap \operatorname{Supp} A_j = \emptyset$; for continuous degrees of membership, strong completeness is violated.

This problem does not occur if $\underset{\cdot}{\wedge}$ has zero divisors (e.g., the Łukasiewicz t-norm).

However, this choice may easily violate the strong completeness [Moser, Navara 1999].

Correctness of residuum-based controller

Theorem: $\forall j : \Phi_{\mathsf{RES}}(A_j) \leq C_j$.

Proof: $X := A_j$,

$$\begin{aligned} \Phi_{\mathsf{RES}}(A_j)(y) &= \sup_x \big(A_j(x) \underset{\cdot}{\wedge} \min_i (A_i(x) \underset{\cdot}{\rightarrow} C_i(y))\big) \\ &\leq \sup_x \big(A_j(x) \underset{\cdot}{\wedge} (A_j(x) \underset{\cdot}{\rightarrow} C_j(y))\big) \leq C_j(y) \,. \end{aligned}$$

Theorem: If there is a fuzzy relation R such that $\forall j : A_j \underset{\cdot}{\circ} R = C_j$, then also R_{RES} satisfies these equalities (and it is the largest solution).

Proof: $\forall j \; \forall x \; \forall y$:

$$\begin{aligned} A_j(x) \underset{\cdot}{\wedge} R(x,y) &\leq C_j(y) \\ R(x,y) &\leq A_j(x) \underset{\cdot}{\rightarrow} C_j(y) \\ R(x,y) &\leq \min_i \big(A_i(x) \underset{\cdot}{\rightarrow} C_i(y)\big) = R_{\mathsf{RES}}(x,y) \,, \end{aligned}$$

$$C_j = A_j \underset{\cdot}{\circ} R \leq A_j \underset{\cdot}{\circ} R_{\mathsf{RES}} \leq C_j \,.$$

What happens if correctness is violated?

Nothing serious, this is usually accepted and possibly compensated during the tuning.

However, it causes a distorted interpretation of (possibly good) control rules.

Initial rule base

Can be obtained by

- asking an expert,
- observing him/her at work,
- combination with analysis of a model (if available),
- a template for a similar problem.

Automatic derivation of rules can be made by clustering methods in the space $\mathcal{X} \times \mathcal{Y}$.
The clusters are approximated by cylindrical extensions of antecedents and consequents.

Tuning

In the phase of tuning, we may

- modify membership functions of antecedents and consequents,
- add new rules,
- delete irrelevant rules or join them with similar ones,

by

- experimenting with the controller,
- observing a human controlling the system (interpretability is needed),

using

- neural networks,
- genetic algorithms, etc.

Requirements on defuzzification

- Continuity.
- Disambiguity.
- Small computational complexity.
- Plausibility (the resulting value should be approximately in the middle of the support and have a high degree of membership).
- Weight counting? (When several firing rules have the same consequent, should we sum them up?)

Methods of defuzzification

- Center of area (gravity) – ignores the multiplicity of overlapping consequents.
  - Continuity: excellent.
  - Disambiguity: none.
  - Computational complexity: high.
  - Plausibility: doubtful! (it may choose a wrong value between two peeks).

Methods of defuzzification

- Center of area (gravity) – ignores the multiplicity of overlapping consequents.
- Center of sums – respects the multiplicity of overlapping consequents.
  - Continuity: excellent.
  - Disambiguity: none.
  - Computational complexity: moderate (centroids corresponding to separate rules may sometimes be computed in advance).
  - Plausibility: doubtful! (it may choose a wrong value between two peeks).

Methods of defuzzification

- Center of area (gravity) – ignores the multiplicity of overlapping consequents.
- Center of sums – respects the multiplicity of overlapping consequents.
- Center of largest area.
  - Continuity: sometimes violated.
  - Disambiguity: sometimes violated.
  - Computational complexity: moderate.
  - Plausibility: reasonable.

Methods of defuzzification

- Center of area (gravity) – ignores the multiplicity of overlapping consequents.
- Center of sums – respects the multiplicity of overlapping consequents.
- Center of largest area;
- First/last of maxima.
  - Continuity: bad!
  - Disambiguity: only due to an additional criterion.
  - Computational complexity: low.
  - Plausibility: reasonable.

Methods of defuzzification

- Center of area (gravity) – ignores the multiplicity of overlapping consequents.
- Center of sums – respects the multiplicity of overlapping consequents.
- Center of largest area.
- First/last of maxima.
- Middle of maxima.
  - Continuity: bad!
  - Disambiguity: only due to an additional criterion.
  - Computational complexity: low.
  - Plausibility: may be a problem.

Methods of defuzzification

- Center of area (gravity) – ignores the multiplicity of overlapping consequents.
- Center of sums – respects the multiplicity of overlapping consequents;
- Center of largest area.
- First/last of maxima.
- Middle of maxima.
- Any of maxima (chosen at random).
  - Continuity: bad!
  - Disambiguity: sometimes violated!
  - Computational complexity: low.
  - Plausibility: reasonable.
  - Can be applied to any form of consequents (not necessarily convex or even non-numerical).

Methods of defuzzification

- Center of area (gravity) – ignores the multiplicity of overlapping consequents.
- Center of sums – respects the multiplicity of overlapping consequents.
- Center of largest area.
- First/last of maxima.
- Middle of maxima.
- Any of maxima (chosen at random).
- Height defuzzification (each consequent is replaced by a singleton and their weighted mean is computed).
  - Continuity: good.
  - Disambiguity: none.
  - Computational complexity: low.
  - Plausibility: doubtful!.
  - Some features of fuzzy control are lost; in fact, crisp outputs of rules are combined.

Methods of defuzzification

- Center of area (gravity) – ignores the multiplicity of overlapping consequents.
- Center of sums – respects the multiplicity of overlapping consequents.
- Center of largest area.
- First/last of maxima.
- Middle of maxima.
- Any of maxima (chosen at random).
- Height defuzzification (each consequent is replaced by a singleton and their weighted mean is computed).

Defuzzification

Problems of defuzzification:

- Multiple maxima.
- Continuous switching between rules.
- If supports of consequents are not bounded, extending the universe may lead to different outputs.

Takagi–Sugeno controller

Uses rules in a generalized form

if X is A_1 **then** Y is $f_1(X)$ **and**

$\ldots$

if X is A_n **then** Y is $f_n(X)$,

where f_i, $i = 1, \ldots, n$, may be arbitrary functions of the input variables (usually linear).

In particular, we may choose any classical controller for f_i.

The advantage is that we added the assumptions of applicability of different rules; as these assumptions are fuzzy, we may switch smoothly from one rule to another.

The output is usually a linear combination (weighted mean) or other aggregation operator applied to the separate rules and taking into account the degrees of firing of the rules.

Defuzzification is not necessary.

Evaluation of fuzzy control in comparison to the classical control

Problems:

- It is difficult to guarantee some properties, in particular stability.
- Number of rules (**curse of dimensionality**).
- Adding new rules, the output of a Mamdani–Assilian controller increases, that of a residuum-based controller decreases; in both cases, they may degenerate.
  In CFR controller this negative effect is compensated, the influence of old rules is attenuated when new rule applies (this is caused by the formula for the degree of conditional firing).

Evaluation of fuzzy control in comparison to the classical control

Advantages:

- Easy design and tuning.
- Simplicity and fast action.
- Interpretability (before/after tuning).
- Possible combination of a theoretical model, automatic generation of rules, and human expertise.
- **Universal approximation property**: For each continuous function on a compact universe and for each ε there is a fuzzy controller which ε-approximates the given function.

  However, the number of rules is not bounded (like in the Weierstrass theorem).

Other areas of application

Any form of approximation, also in computer vision.

Decision making – CFR tested in [Peri 2003, Navara, Peri 2004] as an extension of **FURL (Fuzzy Rule Learner)**, [Yager et al. 2002] in medical diagnostic systems.

Expert systems.

Human-machine interface.

Intelligent database search (Google).

Any field which needs to represent linguistic knowledge in a program.

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