Exercise from properties of fuzzy logic 1

Exercise 6.1 *Verify that the interpretation of*

$$A \underset{\mathrm{S}}{\wedge} B = A \wedge (A \to B)$$

$$A \overset{\mathrm{S}}{\vee} B = ((A \to B) \to B) \underset{\mathrm{S}}{\wedge} ((B \to A) \to A)$$

in basic logic is min, max *for any choice of the continuous fuzzy conjunction.*

Solution: For an evaluation e, we have to prove that the values $\alpha = e(A)$, $\beta = e(B)$, satisfy

$$\alpha \underset{\mathrm{S}}{\wedge} \beta = \alpha \underset{\cdot}{\wedge} (\alpha \underset{\cdot}{\to} \beta) \tag{1}$$

$$\alpha \overset{\mathrm{S}}{\vee} \beta = ((\alpha \underset{\cdot}{\to} \beta) \underset{\cdot}{\to} \beta) \underset{\mathrm{S}}{\wedge} ((\beta \underset{\cdot}{\to} \alpha) \underset{\cdot}{\to} \alpha) \tag{2}$$

(1):
Case 1: $\alpha \leq \beta$.
Trivial, value α.

Case 2: $\alpha > \beta$.
As the fuzzy conjunction $\underset{\cdot}{\wedge}$ is continuous, the supremum

$$\alpha \underset{\cdot}{\to} \beta = \sup\{\gamma : \alpha \underset{\cdot}{\wedge} \gamma \leq \beta\}$$

is a maximum, achieved for some γ^*, and $\alpha \underset{\cdot}{\wedge} \gamma^* = \beta$. The whole expression is β.

(2):
The formula is symmetric w.r.t. α, β. Without loss of generality, we may assume $\alpha \geq \beta$. Then

$$(\beta \underset{\cdot}{\to} \alpha) \underset{\cdot}{\to} \alpha = 1 \underset{\cdot}{\to} \alpha = \alpha\,.$$

It remains to prove that $(\alpha \underset{\cdot}{\to} \beta) \underset{\cdot}{\to} \beta \geq \alpha$. Again, $\alpha \underset{\cdot}{\to} \beta$ is a maximal γ^* such that $\alpha \underset{\cdot}{\wedge} \gamma^* = \beta$.

$$(\alpha \underset{\cdot}{\to} \beta) \underset{\cdot}{\to} \beta = \gamma^* \underset{\cdot}{\to} \beta = \max\{\delta : \gamma^* \underset{\cdot}{\wedge} \delta \leq \beta\} \geq \alpha\,.$$

The last inequality follows from the fact that α belongs to the set over which we maximize. Notice that equality is not always obtained; it may happen that $\gamma^* = \beta$ and $\gamma^* \underset{\cdot}{\to} \beta = 1 > \alpha$. This is the case of Gödel logic (standard fuzzy operations) if $\alpha > \beta$ and also product logic (product operations) if $\alpha > \beta = 0$.