Mathematics 6F – Fuzzy Sets

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http://cmp.felk.cvut.cz/~navara/m6f/fset_printE.pdf

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1 The notion of fuzzy set

1.1 Minimum about (classical) sets

To aviod problems of the set theory, we restrict ourselves to subsets of some **universal set** (**universe**) X

$\mathcal{P}(X)$ denotes the set of all subsets of a set X

A set $A \in \mathcal{P}(X)$ is uniquely determined by its **characteristic function** (**indicator**) $\mu_A : X \to \{0, 1\}$,

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

$$A = \{x \in X : \ \mu_A(x) = 1\} = \{x \in X : \ \mu_A(x) > 0\}.$$

Using the notation

$$\mu_A^{-1}(M) = \{x \in X : \ \mu_A(x) \in M\},$$

we may write

$$A = \mu_A^{-1}\big(\{1\}\big) = \mu_A^{-1}\big((0, 1]\big).$$

Instead of $\mu_A^{-1}\big(\{1\}\big)$, we write $\mu_A^{-1}(1)$, etc.

In particular $\mu_\emptyset = 0$, $\mu_X = 1$.

1.2 Definition of fuzzy sets

A **fuzzy subset** of a universe X (a **fuzzy set**) is a mathematical object A described by its (generalized) **characteristic function** (**membership function**) $\mu_A : X \to [0, 1]$

Alternative notation: $A(x)$

In this context, "classical" sets are called **crisp** or **sharp**.

$\mathcal{F}(X)$ denotes the set of all fuzzy subsets of a universe X

Range (**level set**): $\mathrm{Range}(A) = \big\{\alpha \in [0, 1] : \ (\exists x \in X : \ \mu_A(x) = \alpha)\big\} = \mu_A(X)$

Height: $h(A) = \sup \mathrm{Range}(A)$

Support: $\mathrm{Supp}(A) = \big\{x \in X : \ \mu_A(x) > 0\big\} = \mu_A^{-1}\big((0, 1]\big)$

Core: $\mathrm{core}(A) = \big\{x \in X : \ \mu_A(x) = 1\big\} = \mu_A^{-1}(1)$

2 Examples of fuzzy sets

$A, B \in \mathcal{F}(\mathbf{R})$,

$$\mu_A(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \in [0, 1], \\ 2 - x & \text{if } x \in (1, 2], \\ 0 & \text{if } x > 2, \end{cases}$$

$$\mu_B(x) = \begin{cases} \frac{1}{2} & \text{if } x = 3, \\ 1 & \text{if } x = 4, \\ \frac{1}{4} & \text{if } x = 5, \\ 0 & \text{otherwise.} \end{cases}$$

For finite fuzzy sets, we use an abbreviated notation like $\mu_B = \{(3, \frac{1}{2}), (4, 1), (5, \frac{1}{4})\}$.

Alternative notations: $\mu_B = \{\frac{1}{2}/3, 1/4, \frac{1}{4}/5\}$, $\mu_B = \frac{1}{2}/3 + 1/4 + \frac{1}{4}/5$.

3 System of cuts of a fuzzy set

Definition: Let $A \in \mathcal{F}(X),\ \alpha \in [0,1]$. The **$\alpha$-level** of A is the crisp set

$$\mu_A^{-1}(\alpha) = \{x \in X : \ \mu_A(x) = \alpha\}.$$

The **system of cuts** of A is the mapping $\mathcal{R}_A : [0,1] \to \mathcal{P}(X)$ which assigns to each $\alpha \in [0,1]$ the **α-cut**

$$\mathcal{R}_A(\alpha) = \mu_A^{-1}\big([\alpha,1]\big) = \{x \in X : \ \mu_A(x) \geq \alpha\}.$$

The **system of strong cuts** is the mapping $\mathcal{S}_A : [0,1] \to \mathcal{P}(X)$, where

$$\mathcal{S}_A(\alpha) = \mu_A^{-1}\big((\alpha,1]\big) = \{x \in X : \ \mu_A(x) > \alpha\}.$$

Alternative notations of α-cuts: $[A]_\alpha$, $[A]^\alpha$, ${}^\alpha A$, ${}_\alpha A$

$$\begin{aligned}
\mathrm{Range}(A) &= \{\alpha \in [0,1] : \ \mu_A^{-1}(\alpha) \neq \emptyset\},\\
h(A) &= \sup\{\alpha \in [0,1] : \ \mathcal{R}_A(\alpha) \neq \emptyset\},\\
\mathrm{Supp}(A) &= \mathcal{S}_A(0),\\
\mathrm{core}(A) &= \mathcal{R}_A(1),\\
\mathcal{R}_A(0) &= X,\\
\mathcal{S}_A(1) &= \emptyset.
\end{aligned}$$

4 The first representation theorem

Theorem: A mapping $M : [0,1] \to \mathcal{P}(X)$ is the system of cuts of some fuzzy set $A \in \mathcal{F}(X)$ if and only if
(R1) $M(0) = X$,
(R2) $0 \leq \alpha < \beta \leq 1 \Rightarrow M(\alpha) \supseteq M(\beta)$,
(R3) $0 < \beta \leq 1 \Rightarrow M(\beta) = \bigcap_{\alpha:\alpha<\beta} M(\alpha)$.

Proof:
'$\Rightarrow$': (R1): $M(0) = \mathcal{R}_A(0) = X$.
(R2): $x \in M(\beta) = \mathcal{R}_A(\beta) \Rightarrow \mu_A(x) \geq \beta > \alpha \Rightarrow x \in \mathcal{R}_A(\alpha) = M(\alpha)$.
(R3) '$\subseteq$': (R2) $\Rightarrow \forall\alpha \in [0,\beta) : M(\beta) \subseteq M(\alpha) \Rightarrow M(\beta) \subseteq \bigcap_{\alpha:\alpha<\beta} M(\alpha)$.

(R3) '$\supseteq$': $x \in \bigcap_{\alpha:\alpha<\beta} M(\alpha) = \bigcap_{\alpha:\alpha<\beta} \mathcal{R}_A(\alpha) \Rightarrow \forall\alpha \in [0,\beta) : \mu_A(x) \geq \alpha$,
$\Rightarrow \mu_A(x) \geq \beta \iff x \in \mathcal{R}_A(\beta) = M(\beta)$.

'$\Leftarrow$': We shall prove that $M = \mathcal{R}_A$, where $\mu_A(x) := \sup\{\alpha \in [0,1] : \ x \in M(\alpha)\}$.
'$\subseteq$': $x \in M(\beta) \Rightarrow \mu_A(x) \geq \beta \iff x \in \mathcal{R}_A(\beta)$,
'$\supseteq$': $x \in \mathcal{R}_A(\beta) \Rightarrow \mu_A(x) = \sup\{\alpha \in [0,1] : \ x \in M(\alpha)\} \geq \beta$,
$\forall\alpha \in [0,\beta) : x \in M(\alpha)$,
$x \in \bigcap_{\alpha:\alpha<\beta} M(\alpha) = M(\beta)$.

5 Representations of fuzzy set

Horizontal representation: system of cuts
Vertical representation: membership function
Conversion from the horizontal to vertical representation:

$$\mu_A(x) = \sup\{\alpha \in [0,1] : \ x \in \mathcal{R}_A(\alpha)\}.$$

Theorem: (**the second representation theorem**) Let $A \in \mathcal{F}(X)$. Then

$$\mu_A = \sup_{\alpha\in[0,1]} \alpha\, \mu_{\mathcal{R}_A(\alpha)} = \sup_{\alpha\in\mathrm{Range}(A)} \alpha\, \mu_{\mathcal{R}_A(\alpha)},$$

where the supremum is computed pointwise, i.e.,

$$\mu_A(x) = \sup_{\alpha\in\mathrm{Range}(A)} \alpha\, \mu_{\mathcal{R}_A(\alpha)}(x).$$

5.1 Fuzzy inclusion

Classical definition
$A \subseteq B \iff \forall x \in A:\ x \in B$
cannot be used, because we cannot write $x \in A, x \in B$

However, we can write
$A \subseteq B \iff \forall x \in X:\ \mu_A(x) \le \mu_B(x) \iff \mu_A \le \mu_B$

For $A,\, B \in \mathcal{F}(X)$:
$A \subseteq B \iff \forall x \in X:\ \mu_A(x) \le \mu_B(x) \iff \mu_A \le \mu_B \iff$
$\forall \alpha \in [0,1]:\ \mathcal{R}_A(\alpha) \subseteq \mathcal{R}_B(\alpha)$

Proof of the last equivalence:
'$\Rightarrow$': Assume $\mu_A \le \mu_B$, $x \in \mathcal{R}_A(\alpha)$,
$\alpha \le \mu_A(x) \le \mu_B(x)$, $x \in \mathcal{R}_B(\alpha)$, i.e., $\mathcal{R}_A(\alpha) \subseteq \mathcal{R}_B(\alpha)$
'$\Leftarrow$': Assume $\forall \alpha \in [0,1]:\ \mathcal{R}_A(\alpha) \subseteq \mathcal{R}_B(\alpha)$,
$\mu_A(x) = \sup\{\alpha \in [0,1]:\ x \in \mathcal{R}_A(\alpha)\} \le \sup\{\alpha \in [0,1]:\ x \in \mathcal{R}_B(\alpha)\} = \mu_B(x)$

5.2 Cut-consistency

A **property** P of fuzzy sets $A_1,\dots,A_n$ maps arguments $A_1,\dots,A_n$ to a truth value $P(A_1,\dots,A_n) \in \{0,1\}$ ("predicate").

Property P of of fuzzy sets is called

- **cutworthy** if
$$P(A_1,\dots,A_n) \Rightarrow (\forall \alpha \in (0,1] : P(\mathcal{R}_{A_1}(\alpha),\dots,\mathcal{R}_{A_n}(\alpha)))\,,$$

- **cut-consistent** if
$$P(A_1,\dots,A_n) \iff (\forall \alpha \in (0,1] : P(\mathcal{R}_{A_1}(\alpha),\dots,\mathcal{R}_{A_n}(\alpha)))\,.$$

(0-cuts are ignored intentionally)
Examples:
Inclusion is cut-consistent.
Strong normality, $\exists x \in X:\ \mu_A(x) = 1$, is cut-consistent.
Crispness is cutworthy, but not cut-consistent.

6 Operations with fuzzy sets

6.1 Operations with crisp sets

set operations	propositional operations	formula
$\overline{} : \mathcal{P}(X) \to \mathcal{P}(X)$	$\neg : \{0,1\} \to \{0,1\}$	$\overline{A} = \{x \in X:\ \neg(x \in A)\}$
$\cap : \mathcal{P}(X)^2 \to \mathcal{P}(X)$	$\wedge : \{0,1\}^2 \to \{0,1\}$	$A \cap B = \{x \in X:\ (x \in A) \wedge (x \in B)\}$
$\cup : \mathcal{P}(X)^2 \to \mathcal{P}(X)$	$\vee : \{0,1\}^2 \to \{0,1\}$	$A \cup B = \{x \in X:\ (x \in A) \vee (x \in B)\}$

By means of membership functions:
$\mu_{\overline{A}}(x) = \neg\mu_A(x)$
$\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$
$\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$

6.2 Laws of Boolean algebras

$$
\begin{array}{rcl@{\qquad}rcl}
\neg\neg\alpha & = & \alpha, & & & \\
\alpha \vee \beta & = & \beta \vee \alpha, & \alpha \wedge \beta & = & \beta \wedge \alpha, \\
(\alpha \vee \beta) \vee \gamma & = & \alpha \vee (\beta \vee \gamma), & (\alpha \wedge \beta) \wedge \gamma & = & \alpha \wedge (\beta \wedge \gamma), \\
\alpha \wedge (\beta \vee \gamma) & = & (\alpha \wedge \beta) \vee (\alpha \wedge \gamma), & \alpha \vee (\beta \wedge \gamma) & = & (\alpha \vee \beta) \wedge (\alpha \vee \gamma), \\
\alpha \vee \alpha & = & \alpha, & \alpha \wedge \alpha & = & \alpha, \\
\alpha \vee (\alpha \wedge \beta) & = & \alpha, & \alpha \wedge (\alpha \vee \beta) & = & \alpha, \\
\alpha \vee 1 & = & 1, & \alpha \wedge 0 & = & 0, \\
\alpha \vee 0 & = & \alpha, & \alpha \wedge 1 & = & \alpha, \\
\alpha \wedge \neg\alpha & = & 0, & \alpha \vee \neg\alpha & = & 1, \\
\neg(\alpha \wedge \beta) & = & \neg\alpha \vee \neg\beta, & \neg(\alpha \vee \beta) & = & \neg\alpha \wedge \neg\beta.
\end{array}
$$

6.3 Fuzzy negation

unary operation $\underset{\cdot}{\neg} : [0,1] \to [0,1]$ such that

$$\alpha \le \beta \Rightarrow \underset{\cdot}{\neg}\beta \le \underset{\cdot}{\neg}\alpha, \tag{N1}$$

$$\underset{\cdot}{\neg}\,\underset{\cdot}{\neg}\alpha = \alpha. \tag{N2}$$

Example: Standard negation: $\underset{s}{\neg}\alpha = 1 - \alpha$.

Properties of fuzzy negations

Theorem: Each fuzzy negation $\underset{\cdot}{\neg}$ is a continuous, strictly decreasing bijection satisfying

$$\underset{\cdot}{\neg} 1 = 0, \qquad \underset{\cdot}{\neg} 0 = 1. \tag{N0}$$

Its graph is symmetric w.r.t. the axis of the 1st and 3rd quadrant, i.e., $\underset{\cdot}{\neg}^{-1} = \underset{\cdot}{\neg}$

Proof:

- Injectivity: If $\underset{\cdot}{\neg}\alpha = \underset{\cdot}{\neg}\beta$, then $\alpha = \underset{\cdot}{\neg}\,\underset{\cdot}{\neg}\alpha = \underset{\cdot}{\neg}\,\underset{\cdot}{\neg}\beta = \beta$.
- Surjectivity: For each $\alpha \in [0,1]$ there is a $\beta \in [0,1]$ such that $\alpha = \underset{\cdot}{\neg}\beta$, namely $\beta = \underset{\cdot}{\neg}\alpha$.
- $\Rightarrow$ continuity and boundary conditions.
- The symmetry of the graph is equivalent to involutivity (N2).

Representation theorem for fuzzy negations

A function $\underset{\cdot}{\neg} : [0,1] \to [0,1]$ is a fuzzy negation iff there is an increasing bijection $i : [0,1] \to [0,1]$ (**generator of fuzzy negation** $\underset{\cdot}{\neg}$) such that

$$\underset{\cdot}{\neg} = i \circ \underset{s}{\neg} \circ i^{-1}, \quad \text{i.e.,} \quad \underset{\cdot}{\neg}\alpha = i^{-1}\big(\underset{s}{\neg}\, i(\alpha)\big).$$

Proof: (According to [Nguyen-Walker].)

- Sufficiency:

(N1): Assume $\alpha, \beta \in [0,1]$, $\alpha \le \beta$.
i, i^{-1} preserve the ordering, $\underset{s}{\neg}$ reverses it:

$$
\begin{array}{rcl}
i(\alpha) & \le & i(\beta) \\
\underset{s}{\neg}\, i(\alpha) & \ge & \underset{s}{\neg}\, i(\beta) \\
i^{-1}\big(\underset{s}{\neg}\, i(\alpha)\big) & \ge & i^{-1}\big(\underset{s}{\neg}\, i(\beta)\big) \\
\underset{\cdot}{\neg}\alpha & \ge & \underset{\cdot}{\neg}\beta
\end{array}
$$

(N2): $\underset{\cdot}{\neg} \circ \underset{\cdot}{\neg} = i \circ \underset{s}{\neg} \circ i^{-1} \circ i \circ \underset{s}{\neg} \circ i^{-1} = i \circ \underset{s}{\neg} \circ \underset{s}{\neg} \circ i^{-1} = i \circ i^{-1} = \mathrm{id}$,
where id is the identity on $[0,1]$.

Possible construction of a generator of a fuzzy negation

- Necessity: We shall prove that

$$i(\alpha) = \frac{\alpha + \underset{S}{\neg} \underset{\cdot}{\neg} \alpha}{2}$$

is a generator of a fuzzy negation $\underset{\cdot}{\neg}$.
i is increasing, continuous, and satisfies $i(0) = 0$, $i(1) = 1$, thus i is a bijection on $[0, 1]$.

$$\begin{aligned}
\underset{S}{\neg} i(\alpha) &= 1 - \frac{\alpha + \underset{S}{\neg} \underset{\cdot}{\neg} \alpha}{2} = \frac{1 - \alpha + 1 - \underset{S}{\neg} \underset{\cdot}{\neg} \alpha}{2} = \frac{\underset{S}{\neg} \alpha + \underset{S}{\neg} \underset{S}{\neg} \underset{\cdot}{\neg} \alpha}{2} = \\
&= \frac{\underset{S}{\neg} \alpha + \underset{\cdot}{\neg} \alpha}{2} = \frac{\underset{S}{\neg} \underset{\cdot}{\neg} \underset{\cdot}{\neg} \alpha + \underset{\cdot}{\neg} \alpha}{2} = i(\underset{\cdot}{\neg} \alpha). \\
i \circ \underset{S}{\neg} &= \underset{\cdot}{\neg} \circ i, \text{ i.e., } i \circ \underset{S}{\neg} \circ i^{-1} = \underset{\cdot}{\neg}
\end{aligned}$$

A generator of a fuzzy negation is not unique.

6.4 Fuzzy complement

$$\mu_{\overline{A}^{\cdot}}(x) = \underset{\cdot}{\neg} \mu_A(x).$$

We distinguish them by the same indices as the corresponding fuzzy negations, e.g., $\overline{A}^S$ is the standard complement.

6.5 Fuzzy conjunction (triangular norm, t-norm)

binary operation $\underset{\cdot}{\wedge} : [0, 1]^2 \to [0, 1]$ such that, for all $\alpha, \beta, \gamma \in [0, 1]$:

$$\alpha \underset{\cdot}{\wedge} \beta = \beta \underset{\cdot}{\wedge} \alpha \qquad \text{(commutativity)} \tag{T1}$$

$$\alpha \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\wedge} \gamma) = (\alpha \underset{\cdot}{\wedge} \beta) \underset{\cdot}{\wedge} \gamma \qquad \text{(associativity)} \tag{T2}$$

$$\beta \le \gamma \Rightarrow \alpha \underset{\cdot}{\wedge} \beta \le \alpha \underset{\cdot}{\wedge} \gamma \qquad \text{(monotony)} \tag{T3}$$

$$\alpha \underset{\cdot}{\wedge} 1 = \alpha \qquad \text{(boundary condition)} \tag{T4}$$

Theorem: $\alpha \underset{\cdot}{\wedge} 0 = 0$.

Proof: Using (T3) and (T4): $\alpha \underset{\cdot}{\wedge} 0 \overset{\text{(T3)}}{\le} 1 \underset{\cdot}{\wedge} 0 \overset{\text{(T4)}}{=} 0$.

Examples of fuzzy conjunctions

- **Standard** conjunction (**min**, **Gödel**, **Zadeh**, ...):

$$\alpha \underset{S}{\wedge} \beta = \min(\alpha, \beta).$$

- **Łukasiewicz** conjunction (**Giles**, **bold**, ...):

$$\alpha \underset{L}{\wedge} \beta = \begin{cases} \alpha + \beta - 1 & \text{if } \alpha + \beta - 1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

- **Product** conjunction (**probabilistic**, **Goguen**, **algebraic product**, ...):

$$\alpha \underset{P}{\wedge} \beta = \alpha \cdot \beta.$$

- **Drastic** conjunction (**weak**, ...):

$$\alpha \underset{D}{\wedge} \beta = \begin{cases} \alpha & \text{if } \beta = 1, \\ \beta & \text{if } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Properties of fuzzy conjunctions

Theorem:

$$\forall \alpha, \beta \in [0,1]: \quad \alpha \underset{D}{\wedge} \beta \leq \alpha \underset{\cdot}{\wedge} \beta \leq \alpha \underset{S}{\wedge} \beta.$$

Proof: If $\alpha = 1$ or $\beta = 1$, then (T4) gives the same result for all fuzzy conjunctions. Assume (without loss of generality) that $\alpha \leq \beta < 1$. Then

$$\alpha \underset{D}{\wedge} \beta = 0 \leq \alpha \underset{\cdot}{\wedge} \beta \leq \alpha \underset{\cdot}{\wedge} 1 = \alpha = \alpha \underset{S}{\wedge} \beta.$$

Theorem: Standard conjunction is the only one which is **idempotent**, i.e., $\forall \alpha \in [0,1] : \alpha \underset{\cdot}{\wedge} \alpha = \alpha$

Proof: Assume $\alpha, \beta \in [0,1]$, $\alpha \leq \beta$.

$$\alpha = \alpha \underset{\cdot}{\wedge} \alpha \overset{(T3)}{\leq} \alpha \underset{\cdot}{\wedge} \beta \overset{(T3)}{\leq} \alpha \underset{\cdot}{\wedge} 1 \overset{(T4)}{=} \alpha,$$

thus $\alpha \underset{\cdot}{\wedge} \beta = \alpha = \alpha \underset{S}{\wedge} \beta$.
Analogously for $\alpha > \beta$.

Reprezentation of fuzzy conjunctions (in general)

Theorem: Let $\underset{1}{\wedge}$ be a fuzzy conjunction and $i : [0,1] \to [0,1]$ be an increasing bijection. Then the operation $\underset{2}{\wedge} : [0,1]^2 \to [0,1]$ defined by

$$\alpha \underset{2}{\wedge} \beta = i^{-1}\big(i(\alpha) \underset{1}{\wedge} i(\beta)\big)$$

is a fuzzy conjunction. If $\underset{1}{\wedge}$ is continuous, so is $\underset{2}{\wedge}$.

Proof:

- Commutativity (analogously for associativity):

$$\alpha \underset{2}{\wedge} \beta = i^{-1}(i(\alpha) \underset{1}{\wedge} i(\beta)) = i^{-1}(i(\beta) \underset{1}{\wedge} i(\alpha)) = \beta \underset{2}{\wedge} \alpha$$

- Monotony: Assume $\beta \leq \gamma$.

$$\begin{aligned} i(\beta) &\leq i(\gamma), \\ i(\alpha) \underset{1}{\wedge} i(\beta) &\leq i(\alpha) \underset{1}{\wedge} i(\gamma), \\ \alpha \underset{2}{\wedge} \beta = i^{-1}(i(\alpha) \underset{1}{\wedge} i(\beta)) &\leq i^{-1}(i(\alpha) \underset{1}{\wedge} i(\gamma)) = \alpha \underset{2}{\wedge} \gamma. \end{aligned}$$

- Boundary condition:

$$\alpha \underset{2}{\wedge} 1 = i^{-1}(i(\alpha) \underset{1}{\wedge} i(1)) = i^{-1}(i(\alpha) \underset{1}{\wedge} 1) = i^{-1}(i(\alpha)) = \alpha.$$

Classification of fuzzy conjunctions

Continuous fuzzy conjunction $\underset{\cdot}{\wedge}$ is

- **Archimedean** if

$$\forall \alpha \in (0,1): \quad \alpha \underset{\cdot}{\wedge} \alpha < \alpha \tag{TA}$$

- **strict** if

$$\forall \alpha \in (0,1]\ \forall \beta, \gamma \in [0,1]: \quad \beta < \gamma \Rightarrow \alpha \underset{\cdot}{\wedge} \beta < \alpha \underset{\cdot}{\wedge} \gamma \tag{T3+}$$

- **nilpotent** if it is Archimedean and not strict.

Example: Product conjunction is strict, Łukasiewicz conjunction is nilpotent, standard and drastic conjunctions are not Archimedean (the standard one violates (TA), the drastic one is not continuous).

Representation theorem for strict fuzzy conjunctions

Operation $\underset{\cdot}{\wedge} : [0,1]^2 \to [0,1]$ is a strict fuzzy conjunction iff there is an increasing bijection $i : [0,1] \to [0,1]$ (**multiplicative generator**) such that

$$\alpha \underset{\cdot}{\wedge} \beta = i^{-1}\big(i(\alpha) \underset{\mathrm{P}}{\wedge} i(\beta)\big) = i^{-1}\big(i(\alpha) \cdot i(\beta)\big).$$

Sufficiency has been already proved (except for strictness which is easy).
The proof of necessity is much more advanced.

A multiplicative generator of a strict fuzzy conjunction is not unique.

Representation theorem for nilpotent fuzzy conjunctions

Operation $\underset{\cdot}{\wedge} : [0,1]^2 \to [0,1]$ is a **nilpotent** fuzzy conjunction iff there is an increasing bijection $i : [0,1] \to [0,1]$ (**Łukasiewicz generator**) such that

$$\alpha \underset{\cdot}{\wedge} \beta = i^{-1}\big(i(\alpha) \underset{\mathrm{L}}{\wedge} i(\beta)\big).$$

A Łukasiewicz generator of a nilpotent fuzzy conjunction is not unique.

Theorem: Let $\underset{\cdot}{\wedge}$ be a **nilpotent** fuzzy conjunction. Then

$$\forall \alpha \in (0,1)\ \exists n \in \mathbf{N} : \underset{\cdot}{\bigwedge}\nolimits_{k=1}^{n} \alpha = 0$$

Proof: According to the representation theorem, it suffices (without loss of generality) to prove the theorem for the Łukasiewicz conjunction. For a sufficiently large n we obtain

$$\alpha + \sum_{i=2}^{n} (\alpha - 1) \le 0, \qquad \underset{\mathrm{L}}{\bigwedge}\nolimits_{k=1}^{n} \alpha = 0.$$

6.6 Fuzzy intersection

is an operation on fuzzy sets defined using a fuzzy conjunction:

$$\mu_{A \underset{\cdot}{\cap} B}(x) = \mu_A(x) \underset{\cdot}{\wedge} \mu_B(x)$$

(we distinguish them by the same indices as the respective fuzzy conjunctions)

Theorem: The **standard** intersection is cut-consistent.

Proof: 1. Cutworthiness:

$$\begin{aligned}
\mathcal{R}_{A \underset{\mathrm{S}}{\cap} B}(\alpha) &= \{x \in X : \mu_{A \underset{\mathrm{S}}{\cap} B}(x) \ge \alpha\} \\
&= \{x \in X : (\mu_A(x) \ge \alpha) \wedge (\mu_B(x) \ge \alpha)\} \\
&= \{x \in X : \mu_A(x) \ge \alpha\} \cap \{x \in X : \mu_B(x) \ge \alpha\} \\
&= \mathcal{R}_A(\alpha) \cap \mathcal{R}_B(\alpha)
\end{aligned}$$

2. Cuts $\mathcal{R}_A(\alpha) \cap \mathcal{R}_B(\alpha)$ (for all $\alpha \in (0,1]$) determine a unique fuzzy set equal to $A \underset{\mathrm{S}}{\cap} B$.

6.7 Fuzzy disjunction (triangular conorm, t-conorm)

is a binary operation $\dot{\vee} : [0,1]^2 \to [0,1]$ such that

$$\alpha \mathbin{\dot{\vee}} \beta = \beta \mathbin{\dot{\vee}} \alpha \qquad \text{(commutativity) (S1)}$$

$$\alpha \mathbin{\dot{\vee}} (\beta \mathbin{\dot{\vee}} \gamma) = (\alpha \mathbin{\dot{\vee}} \beta) \mathbin{\dot{\vee}} \gamma \qquad \text{(associativity) (S2)}$$

$$\beta \le \gamma \Rightarrow \alpha \mathbin{\dot{\vee}} \beta \le \alpha \mathbin{\dot{\vee}} \gamma \qquad \text{(monotony) (S3)}$$

$$\alpha \mathbin{\dot{\vee}} 0 = \alpha \qquad \text{(boundary condition) (S4)}$$

Theorem: $\alpha \mathbin{\dot{\vee}} 1 = 1$.

Proof: $\alpha \mathbin{\dot{\vee}} 1 \overset{\text{(S3)}}{\ge} 0 \mathbin{\dot{\vee}} 1 \overset{\text{(S4)}}{=} 1$.

Examples of fuzzy disjunctions

- **Standard** (**max**, **Gödel**, **Zadeh** ...):

$$\alpha \overset{\text{S}}{\vee} \beta = \max(\alpha, \beta).$$

- **Łukasiewicz** (**Giles**, **bold**, **bounded sum** ...):

$$\alpha \overset{\text{L}}{\vee} \beta = \begin{cases} \alpha + \beta & \text{for } \alpha + \beta < 1, \\ 1 & \text{otherwise.} \end{cases}$$

- **Product** (**probabilistic** ...):

$$\alpha \overset{\text{P}}{\vee} \beta = \alpha + \beta - \alpha \cdot \beta.$$

- **Drastic** (**weak** ...):

$$\alpha \overset{\text{D}}{\vee} \beta = \begin{cases} \alpha & \text{for } \beta = 0, \\ \beta & \text{for } \alpha = 0, \\ 1 & \text{otherwise.} \end{cases}$$

- **Einstein**

$$\alpha \overset{\text{E}}{\vee} \beta = \frac{\alpha + \beta}{1 + \alpha\beta}$$

Properties of fuzzy disjunctions

$$\forall \alpha, \beta \in [0,1] : \ \alpha \overset{\text{S}}{\vee} \beta \leq \alpha \dot{\vee} \beta \leq \alpha \overset{\text{D}}{\vee} \beta.$$

The standard disjunction is the only one which is idempotent, i.e., $\alpha \dot{\vee} \alpha = \alpha$ for all $\alpha \in [0,1]$.

Duality

Let $\underset{\cdot}{\neg}$ be a fuzzy negation.

A. If $\underset{\cdot}{\wedge}$ is a fuzzy conjunction, then $\alpha \dot{\vee} \beta = \underset{\cdot}{\neg}(\underset{\cdot}{\neg}\,\alpha \underset{\cdot}{\wedge} \underset{\cdot}{\neg}\,\beta)$ is a fuzzy disjunction (**dual** to $\underset{\cdot}{\wedge}$ with respect to $\underset{\cdot}{\neg}$).

B. If $\dot{\vee}$ is a fuzzy disjunction, then $\alpha \underset{\cdot}{\wedge} \beta = \underset{\cdot}{\neg}(\underset{\cdot}{\neg}\,\alpha \dot{\vee} \underset{\cdot}{\neg}\,\beta)$ is a fuzzy conjunction (**dual** to $\dot{\vee}$ with respect to $\underset{\cdot}{\neg}$).

Theorem:

- The **Łukasiewicz** operations $\underset{\text{L}}{\wedge}, \overset{\text{L}}{\vee}$ are dual with respect to the **standard** negation.
- The **product** operations $\underset{\text{P}}{\wedge}, \overset{\text{P}}{\vee}$ are dual with respect to the **standard** negation.
- The **standard** operations $\underset{\text{S}}{\wedge}, \overset{\text{S}}{\vee}$ are dual with respect to **any** fuzzy negation.
- The **drastic** operations $\underset{\text{D}}{\wedge}, \overset{\text{D}}{\vee}$ are dual with respect to **any** fuzzy negation.

Classification of fuzzy disjunctions

A **continuous** fuzzy disjunction $\dot{\vee}$ is

- **Archimedean** if

$$\forall \alpha \in (0,1) : \ \alpha \dot{\vee} \alpha > \alpha \tag{SA}$$

- **strict** if

$$\forall \alpha \in [0,1) \ \forall \beta, \gamma \in [0,1] : \ \beta < \gamma \Rightarrow \alpha \dot{\vee} \beta < \alpha \dot{\vee} \gamma \tag{S3+}$$

- **nilpotent** if it is Archimedean and not strict.

Representation theorems for fuzzy disjunctions

Theorem: An operation $\dot\vee : [0,1]^2 \to [0,1]$ is a **strict** fuzzy disjunction iff there is an increasing bijection $i : [0,1] \to [0,1]$ such that

$$\alpha \dot\vee \beta = i^{-1}\big(i(\alpha) \overset{\text{P}}{\vee} i(\beta)\big).$$

Theorem: An operation $\dot\vee : [0,1]^2 \to [0,1]$ is a **nilpotent** fuzzy disjunction iff there is an increasing bijection $i : [0,1] \to [0,1]$ (**additive generator**) such that

$$\alpha \dot\vee \beta = i^{-1}\big(i(\alpha) \overset{\text{L}}{\vee} i(\beta)\big) = \begin{cases} i^{-1}\big(i(\alpha) + i(\beta)\big) & \text{if } i(\alpha) + i(\beta) \le 1 \\ 1 & \text{otherwise.} \end{cases}$$

6.8 Fuzzy union

is an operation on fuzzy sets defined using a fuzzy disjunction:

$$\mu_{A \dot\cup B}(x) = \mu_A(x) \dot\vee \mu_B(x).$$

(we distinguish them by the same indices as the respective fuzzy disjunctions)

Theorem: The **standard** union is cut-consistent.

6.9 Fuzzy propositional algebras

equations written in **black** always hold
equations written in **red** hold for the standard fuzzy operations, but not for some others
equations written in **blue** hold only for some choices of fuzzy operations (not for the standard ones)

$$\begin{aligned}
\dot\neg\dot\neg\alpha &= \alpha, & & \\
\alpha \dot\vee \beta &= \beta \dot\vee \alpha, & \alpha \dot\wedge \beta &= \beta \dot\wedge \alpha, \\
(\alpha \dot\vee \beta) \dot\vee \gamma &= \alpha \dot\vee (\beta \dot\vee \gamma), & (\alpha \dot\wedge \beta) \dot\wedge \gamma &= \alpha \dot\wedge (\beta \dot\wedge \gamma), \\
\alpha \dot\wedge (\beta \dot\vee \gamma) &= (\alpha \dot\wedge \beta) \dot\vee (\alpha \dot\wedge \gamma), & \alpha \dot\vee (\beta \dot\wedge \gamma) &= (\alpha \dot\vee \beta) \dot\wedge (\alpha \dot\vee \gamma), \\
\alpha \dot\vee \alpha &= \alpha, & \alpha \dot\wedge \alpha &= \alpha, \\
\alpha \dot\vee (\alpha \dot\wedge \beta) &= \alpha, & \alpha \dot\wedge (\alpha \dot\vee \beta) &= \alpha, \\
\alpha \dot\vee 1 &= 1, & \alpha \dot\wedge 0 &= 0, \\
\alpha \dot\vee 0 &= \alpha, & \alpha \dot\wedge 1 &= \alpha, \\
\alpha \dot\wedge \dot\neg\alpha &= \mathbf{0}, & \alpha \dot\vee \dot\neg\alpha &= \mathbf{1}, \\
\dot\neg(\alpha \dot\wedge \beta) &= \dot\neg\alpha \dot\vee \dot\neg\beta, & \dot\neg(\alpha \dot\vee \beta) &= \dot\neg\alpha \dot\wedge \dot\neg\beta.
\end{aligned}$$

6.10 Fuzzy implication

is any operation $\dot\rightarrow : [0,1]^2 \to [0,1]$ which coincides with the classical implication on $\{0,1\}^2$.
We would like to satisfy the following properties, but we do not require them as axioms:

$$\alpha \dot\rightarrow \beta = 1 \Leftarrow \alpha \le \beta, \tag{I1a}$$

$$\alpha \dot\rightarrow \beta = 1 \Rightarrow \alpha \le \beta, \tag{I1b}$$

$$1 \dot\rightarrow \beta = \beta, \tag{I2}$$

$$\dot\rightarrow \text{ is nonincreasing in the first argument and nondecreasing in the second,} \tag{I3}$$

$$\alpha \dot\rightarrow \beta = \underset{\text{S}}{\neg}\beta \dot\rightarrow \underset{\text{S}}{\neg}\alpha, \tag{I4}$$

$$\alpha \dot\rightarrow (\beta \dot\rightarrow \gamma) = \beta \dot\rightarrow (\alpha \dot\rightarrow \gamma), \tag{I5}$$

$$\text{continuity.} \tag{I6}$$

R-implication (residuated fuzzy implication, residuum)

is an operation

$$\alpha \overset{\mathrm{R}}{\underset{\cdot}{\to}} \beta = \sup\{\gamma :\ \alpha \underset{\cdot}{\wedge} \gamma \le \beta\} \tag{RI}$$

where $\underset{\cdot}{\wedge}$ is a fuzzy conjunction

(if $\underset{\cdot}{\wedge}$ is continuous, we may take the maximum instead of the supremum)

Examples of R-implications

- From the standard conjunction $\underset{\mathrm{S}}{\wedge}$ we obtain the **Gödel implication**

$$\alpha \overset{\mathrm{R}}{\underset{\mathrm{S}}{\to}} \beta = \begin{cases} 1 & \text{if } \alpha \le \beta, \\ \beta & \text{otherwise.} \end{cases}$$

It is piecewise linear and continous except for the points (α, α), $\alpha < 1$.

- From the Łukasiewicz conjunction $\underset{\mathrm{L}}{\wedge}$ we obtain the **Łukasiewicz implication**

$$\alpha \overset{\mathrm{R}}{\underset{\mathrm{L}}{\to}} \beta = \begin{cases} 1 & \text{if } \alpha \le \beta, \\ 1 - \alpha + \beta & \text{otherwise.} \end{cases}$$

It is piecewise linear and continous.

- From the product conjunction $\underset{\mathrm{P}}{\wedge}$ we obtain the **Goguen** (also **Gaines**) **implication**

$$\alpha \overset{\mathrm{R}}{\underset{\mathrm{P}}{\to}} \beta = \begin{cases} 1 & \text{if } \alpha \le \beta, \\ \frac{\beta}{\alpha} & \text{otherwise.} \end{cases}$$

It has one point of discontinuity, $(0, 0)$.

Properties of R-implications

Theorem: Let $\underset{\cdot}{\wedge}$ be a continuous fuzzy conjunction. Then the R-implication $\overset{\mathrm{R}}{\underset{\cdot}{\to}}$ satisfies (I1a), (I1b), (I2), (I3).

Proof: $\alpha \overset{\mathrm{R}}{\underset{\cdot}{\to}} \beta = \sup \Gamma(\alpha, \beta)$, where

$\Gamma(\alpha, \beta) = \{\gamma :\ \alpha \underset{\cdot}{\wedge} \gamma \le \beta\}$ is an interval containing zero. (Moreover, due to the continuity of $\underset{\cdot}{\wedge}$ the interval is closed.)

(I1a) If $\alpha \le \beta$, then $\Gamma(\alpha, \beta) = [0, 1]$, $\sup \Gamma(\alpha, \beta) = 1$.

(I1b) If $\alpha > \beta$, then $1 \notin \Gamma(\alpha, \beta)$, $\sup \Gamma(\alpha, \beta) < 1$ (from the closedness of $\Gamma(\alpha, \beta)$).

(I2): $1 \overset{\mathrm{R}}{\underset{\cdot}{\to}} \beta = \sup\{\gamma :\ \gamma \le \beta\} = \beta$.

(I3): When α increases, $\Gamma(\alpha, \beta)$ does not increase.
When β increases, $\Gamma(\alpha, \beta)$ does not decrease.

Theorem: A residuated fuzzy implication induced by a **continuous** fuzzy conjunction $\underset{\cdot}{\wedge}$ is continuous iff $\underset{\cdot}{\wedge}$ is nilpotent.

S-implication

is an operation

$$\alpha \overset{\mathrm{S}}{\underset{\cdot}{\to}} \beta = \underset{\mathrm{S}}{\neg} \alpha \mathbin{\dot{\vee}} \beta \tag{SI}$$

where $\dot{\vee}$ is a fuzzy disjunction

Example:

- From the standard disjunction we obtain the **Kleene–Dienes** implication

$$\alpha \overset{\mathrm{S}}{\underset{\mathrm{S}}{\to}} \beta = \max(1 - \alpha, \beta).$$

- From the Łukasiewicz disjunction we obtain the **Łukasiewicz** implication $\overset{\text{S}}{\underset{\text{L}}{\rightarrow}}$ which coincides with the Łukasiewicz residuated implication $\overset{\text{R}}{\underset{\text{L}}{\rightarrow}}$.

Among all fuzzy implications studied here, only residuated implications induced by nilpotent fuzzy conjunctions (e.g., the Łukasiewicz implication) satisfy all properties (I1a),(I1b),(I2)–(I6).

6.11 Fuzzy biimplication (equivalence)

is an operation $\dot{\leftrightarrow}$ usually defined by

$$\alpha \dot{\leftrightarrow} \beta = (\alpha \dot{\rightarrow} \beta) \dot{\wedge} (\beta \dot{\rightarrow} \alpha),$$

where $\dot{\rightarrow}$ is a fuzzy implication and $\dot{\wedge}$ is a fuzzy conjunction
(biimplications are distinguished by the same indices as the respective fuzzy implications)
If $\dot{\rightarrow}$ satisfies (I1a) (e.g., for a residuated implication), at least one of the brackets equals 1, hence the choice of the fuzzy conjunction $\dot{\wedge}$ is irrelevant.

Example: Łukasiewicz biimplication: $\alpha \overset{\text{R}}{\underset{\text{L}}{\leftrightarrow}} \beta = 1 - |\alpha - \beta|$.

7 Fuzzy relations

7.1 Classical relations

A **binary relation** is an $R \subseteq X \times Y$
Inverse relation to R: $R^{-1} \subseteq Y \times X$:

$$R^{-1} = \{(y, x) \in Y \times X : \ (x, y) \in R\}$$

The **composition** of relations $R \subseteq X \times Y$, $S \subseteq Y \times Z$ is $R \circ S \subseteq X \times Z$:

$$R \circ S = \Big\{(x, z) \in X \times Z : \ \big(\exists y \in Y : (x, y) \in R, (y, z) \in S\big)\Big\}$$

Using membership functions:

$$\mu_R : X \times Y \to \{0, 1\}$$
$$\mu_{R^{-1}}(y, x) = \mu_R(x, y)$$
$$\mu_{R \circ S}(x, z) = \max_{y \in Y} \big(\mu_R(x, y) \wedge \mu_S(y, z)\big)$$

7.2 Fuzzy relations

A **fuzzy relation** is $R \in \mathcal{F}(X \times Y)$, $\mu_R : X \times Y \to [0, 1]$
The **inverse relation** to R is $R^{-1} \in \mathcal{F}(Y \times X)$:

$$\forall x \in X \ \forall y \in Y : \mu_{R^{-1}}(y, x) = \mu_R(x, y)$$

The $\dot{\wedge}$**-composition** of relations $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ is $R \dot{\circ} S \in \mathcal{F}(X \times Z)$:

$$\mu_{R \dot{\circ} S}(x, z) = \sup_{y \in Y} \big(\mu_R(x, y) \dot{\wedge} \mu_S(y, z)\big)$$

Theorem The inversion of fuzzy relations is cut-consistent.
Theorem If Y is a finite set, then the standard composition of fuzzy relations $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ is cut-consistent.

7.3 Special crisp relations

$R \subseteq X \times X$ can be:

- **an equality**: $E = \{(x,x) : x \in X\}$,
- **reflexive**: $\forall x \in X : (x,x) \in R$, i.e., $E \subseteq R$,
- **symmetric**: $(x,y) \in R \Rightarrow (y,x) \in R$, i.e., $R = R^{-1}$,
- **antisymmetric**: $\big((x,y) \in R\big) \wedge \big((y,x) \in R\big) \Rightarrow x = y$, i.e., $R \cap R^{-1} \subseteq E$,
- **transitive**: $\big((x,y) \in R\big) \wedge \big((y,z) \in R\big) \Rightarrow (x,z) \in R$, i.e., $R \circ R \subseteq R$,
- **a partial order**: antisymmetric, reflexive, and transitive,
- **an equivalence**: symmetric, reflexive, and transitive.

The membership function of the equality relation, $E \subseteq X \times X$, is the **Kronecker delta**:

$$\mu_E(x,y) = \delta(x,y) = \begin{cases} 1 & \text{for } x = y, \\ 0 & \text{for } x \neq y. \end{cases}$$

7.4 Special fuzzy relations

A fuzzy relation $R \in \mathcal{F}(X \times X)$ can be:

- **reflexive**: $E \subseteq R$,
- **symmetric**: $R = R^{-1}$,
- **$\cdot$-antisymmetric**: $R \underset{\cdot}{\cap} R^{-1} \subseteq E$,
- **$\cdot$-transitive**: $R \underset{\cdot}{\circ} R \subseteq R$,
- **a $\cdot$-partial order**: $\cdot$-antisymmetric, reflexive, and $\cdot$-transitive,
- **an $\cdot$-equivalence**: symmetric, reflexive, and $\cdot$-transitive.

The last four terms depend on the choice of the fuzzy conjunction $\underset{\cdot}{\wedge}$.

Theorem The following properties of fuzzy relations are cut-consistent:

- reflexivity,
- symmetry,
- standard antisymmetry,
- standard transitivity,
- standard partial order,
- standard equivalence.

7.5 Projections of fuzzy relations

The **left (first) projection** of a fuzzy relation $R \in \mathcal{F}(X \times Y)$ is $P_1(R) \in \mathcal{F}(X)$:

$$\mu_{P_1(R)}(x) = \sup_{y \in Y} \mu_R(x,y)$$

The **right (second) projection** of a fuzzy relation $R \in \mathcal{F}(X \times Y)$ is $P_2(R) \in \mathcal{F}(Y)$:

$$\mu_{P_2(R)}(y) = \sup_{x \in X} \mu_R(x,y)$$

Theorem The projections of fuzzy relations are cut-consistent.

7.6 Cylindric extension

(also the **cartesian product**) of fuzzy sets $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ is $A \times B \in \mathcal{F}(X \times Y)$:

$$\mu_{A \times B}(x,y) = \mu_A(x) \underset{S}{\wedge} \mu_B(y)$$

It is the maximal fuzzy relation $R \in \mathcal{F}(X \times Y)$ such that $P_1(R) \subseteq A$ and $P_2(R) \subseteq B$. Equality occurs iff $h(A) = h(B)$.

Theorem

$$P_1(R) \times P_2(R) \supseteq R$$

Theorem The cylindric extension is cut-consistent.

8 Extension principle

8.1 The extension of binary relations to crisp sets

A **mapping** is $R \subseteq X \times Y$:

$$\forall x \in X\ \exists! y = r(x) \in Y : (x, y) \in R$$

A mapping $R \subseteq X \times Y$ corresponds to an $r : X \to Y$ by $(x, y) \in R \iff y = r(x)$, $\quad R = \{(x, r(x)) : x \in X\}$

The **extension** of a relation $R \subseteq X \times Y$ is a mapping $r : \mathcal{P}(X) \to \mathcal{P}(Y)$:

$$r(A) = \{y \in Y : (\exists x \in A : (x, y) \in R)\}$$

Analogously, the extension of the relation $R^{-1} \subseteq Y \times X$ is a mapping $r^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$:

$$r^{-1}(B) = \{x \in X : (\exists y \in B : (x, y) \in R)\}$$

The extensions r and r^{-1} are mappings even if the original relation R was not a mapping. However, they are not mutually inverse.

If, moreover, R is a mapping, then

$$\begin{aligned} r(A) &= \{r(x) : x \in A\} \\ r^{-1}(B) &= \{x \in X : r(x) \in B\} \end{aligned}$$

In particular,

$$r^{-1}(y) = r^{-1}(\{y\}) = \{x \in X : r(x) = y\}$$

Using membership functions:

$$\mu_{r(A)}(y) = \max_{x \in X} \big(\mu_R(x, y) \wedge \mu_A(x)\big)$$

$$\mu_{r^{-1}(B)}(x) = \max_{y \in Y} \big(\mu_R(x, y) \wedge \mu_B(y)\big)$$

8.2 The extension of binary relations to fuzzy sets

The **extension** of a relation $R \subseteq X \times Y$ is a mapping $r : \mathcal{F}(X) \to \mathcal{F}(Y)$:

$$\mu_{r(A)}(y) = \sup_{x \in X} \big(\mu_R(x, y) \underset{S}{\wedge} \mu_A(x)\big) \qquad (A \in \mathcal{F}(X),\ y \in Y)$$

Analogously, the extension of the relation $R^{-1} \subseteq Y \times X$ is a mapping $r^{-1} : \mathcal{F}(Y) \to \mathcal{F}(X)$:

$$\mu_{r^{-1}(B)}(x) = \sup_{y \in Y} \big(\mu_R(x, y) \underset{S}{\wedge} \mu_B(y)\big) \qquad (B \in \mathcal{F}(Y),\ x \in X)$$

As R is a crisp relation, the choice of the fuzzy conjunction $\underset{\cdot}{\wedge}$ is irrelevant:

$$\mu_R(x, y) \underset{\cdot}{\wedge} \mu_A(x) = \begin{cases} \mu_A(x) & \text{for } \mu_R(x, y) = 1 \\ 0 & \text{for } \mu_R(x, y) = 0 \end{cases}$$

Using the extensions $r : \mathcal{P}(X) \to \mathcal{P}(Y)$, $\quad r^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ of relations R, R^{-1} to **crisp** sets, the extensions to **fuzzy** sets can be written as

$$\mu_{r(A)}(y) = \sup_{x \in r^{-1}(y)} \mu_A(x)$$

$$\mu_{r^{-1}(B)}(x) = \sup_{y \in r(x)} \mu_B(y)$$

If, moreover, R is a mapping, then

$$\mu_{r^{-1}(B)}(x) = \mu_B(r(x))$$

If R^{-1} is a mapping, then

$$\mu_{r(A)}(y) = \mu_A(r^{-1}(y))$$

Theorem

$$r\big(\mathcal{R}_A(\alpha)\big) \subseteq \mathcal{R}_{r(A)}(\alpha)$$

If the sets

$$r^{-1}(y) = \{x \in X : (x, y) \in R\}$$

are finite for all $y \in Y$, then the equality holds.

8.3 Convex fuzzy sets

Here L denotes a linear space.

A crisp set $A \subseteq L$ is called **convex** if

$$\forall x, y \in A \; \forall \lambda \in (0,1) : \lambda x + (1-\lambda)\, y \in A$$

Using membership functions:

$$\min\bigl(\mu_A(x), \mu_A(y)\bigr) \leq \mu_A\bigl(\lambda x + (1-\lambda)\, y\bigr)$$

Let X be a crisp convex subset of a linear space.

A fuzzy set $A \in \mathcal{F}(X)$ is called **convex** if

$$\forall x, y \in X \; \forall \lambda \in (0,1) : \quad \mu_A\bigl(\lambda x + (1-\lambda)\, y\bigr) \geq \mu_A(x) \underset{\mathrm{S}}{\wedge} \mu_A(y)$$

Convexity of fuzzy sets has nothing in common with the convexity of its membership function!

Theorem Convexity is cut-consistent.

In particular, a fuzzy set of real numbers is convex iff all its cuts are intervals.

8.4 Fuzzy numbers and fuzzy intervals

A **fuzzy interval** is an $A \in \mathcal{F}(\mathbf{R})$ such that:

- Supp A is a bounded set,
- For all $\alpha \in (0,1]$, the cut $\mathcal{R}_A(\alpha)$ is a closed interval,
- $\mathcal{R}_A(1) \neq \emptyset$ (i.e., $\mathcal{R}_A(1)$ is a nonempty closed interval).

If, moreover, $\mathcal{R}_A(1)$ is a singleton, then A is called a **fuzzy number**.

Fuzzy intervals are convex.

The fuzzy interval **inverse** to a fuzzy interval A is $-A \in \mathcal{F}(\mathbf{R})$:

$$\mu_{-A}(x) = \mu_A(-x)$$

(The extension principle for binary relations applied to the unary minus)

$$\mathcal{R}_{-A}(\alpha) = -\mathcal{R}_A(\alpha)$$

8.5 Binary operations with fuzzy intervals

$\Box \in \{+, -, \cdot, /\}$

$\Box : \mathbf{R}^2 \to \mathbf{R}$ can be understood as a crisp relation

$\Box \subseteq \mathbf{R}^2 \times \mathbf{R}$:

$$\mu_{\Box}\bigl((y,z),x\bigr) = \begin{cases} 1 & \text{for } y \,\Box\, z = x, \\ 0 & \text{otherwise.} \end{cases}$$

This can be extended by the (already introduced) extension principle for **binary relations** to an operation $\mathcal{F}(\mathbf{R}^2) \to \mathcal{F}(\mathbf{R})$; this has to be composed with the cylindric extension $\mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to \mathcal{F}(\mathbf{R}^2)$. We obtain the binary operation $\Box : \mathcal{F}(\mathbf{R}) \times \mathcal{F}(\mathbf{R}) \to \mathcal{F}(\mathbf{R})$.

$$\begin{array}{c} A \in \mathcal{F}(\mathbf{R}), \quad B \in \mathcal{F}(\mathbf{R}) \\ \downarrow \\ A \times B \in \mathcal{F}(\mathbf{R} \times \mathbf{R}) \\ \downarrow \\ A \,\Box\, B = \Box(A \times B) \in \mathcal{F}(\mathbf{R}) \end{array}$$

$$\begin{aligned} \mu_{A \Box B}(x) &= \mu_{\Box(A \times B)}(x) \\ &= \sup_{(y,z) \in \mathbf{R}^2} \bigl(\mu_{A \times B}(y,z) \underset{\mathrm{S}}{\wedge} \mu_{\Box}\bigl((y,z),x\bigr)\bigr) \end{aligned}$$

$$
\begin{aligned}
&= \sup_{(y,z)\in\mathbf{R}^2,\, y\,\Box\, z = x} \mu_{A\times B}(y,z) \\
&= \sup_{(y,z)\in\mathbf{R}^2,\, y\,\Box\, z = x} \left(\mu_A(y) \underset{S}{\wedge} \mu_B(z)\right)
\end{aligned}
$$

In particular, for $\Box = +$:
We compute the supremum of the function $\mu_A(y) \underset{S}{\wedge} \mu_B(z)$ over all $y \in \mathbf{R}, z \in \mathbf{R}$ such that $y + z = x$.
This is the supremum of the function $\mu_A(x-z) \underset{S}{\wedge} \mu_B(z)$ over all $z \in \mathbf{R}$ (because $y + z = x \Rightarrow y = x - z$).

$$
\begin{aligned}
\mu_{A+B}(x) &= \sup_{z\in\mathbf{R}} \big(\mu_A(x-z) \underset{S}{\wedge} \mu_B(z)\big), \\
\mu_{A-B}(x) &= \sup_{z\in\mathbf{R}} \big(\mu_A(x+z) \underset{S}{\wedge} \mu_B(z)\big), \\
\mu_{A\cdot B}(x) &= \sup_{z\in\mathbf{R},\, z\neq 0} \big(\mu_A(x/z) \underset{S}{\wedge} \mu_B(z)\big), \quad x \neq 0, \\
\mu_{A/B}(x) &= \sup_{z\in\mathbf{R}} \big(\mu_A(x\cdot z) \underset{S}{\wedge} \mu_B(z)\big).
\end{aligned}
$$

Only for $\mu_{A\cdot B}(0)$ we have to use the original definition (because of problems with division by zero).
In particular, for crisp intervals $A = [a,b]$, $B = [c,d]$ we obtain the **interval arithmetic**:

$$
\begin{aligned}
[a,b] + [c,d] &= [a+c, b+d], \\
[a,b] - [c,d] &= [a-d, b-c], \\
[a,b] \cdot [c,d] &= \big[\min(ac,ad,bc,bd), \max(ac,ad,bc,bd)\big], \\
[a,b] / [c,d] &= \big[\min(a/c,a/d,b/c,b/d), \max(a/c,a/d,b/c,b/d)\big].
\end{aligned}
$$

The latter equality holds only for $0 \notin [c,d]$.

$$
\mu_{A\Box B}(x) = \max\big\{\mu_A(y) \underset{S}{\wedge} \mu_B(z) :\ y,z \in \mathbf{R}, y\,\Box\, z = x\big\}.
$$

(In case of division we assume $\mu_B(0) = 0$.)
Theorem The addition, subtraction, multiplication, and division of fuzzy intervals is cut-consistent. (In case of division we assume zero membership to the divisor.)
Theorem The addition, subtraction, and multiplication of fuzzy numbers (resp. fuzzy intervals) is a fuzzy number (resp. a fuzzy interval). (The same holds for division unless zero is in the closure of the support of the divisor.)
Any real number $x \in \mathbf{R}$ can be understood as a fuzzy number (represented by a crisp singleton $\{x\}$); we denote it by x.
Theorem Properties of operations with fuzzy intervals:

$$
\begin{aligned}
0 + A &= A, \\
0 \cdot A &= 0, \\
1 \cdot A &= A, \\
A + B &= B + A, \\
A \cdot B &= B \cdot A, \\
A + (B + C) &= (A + B) + C, \\
A \cdot (B \cdot C) &= (A \cdot B) \cdot C, \\
A + (-B) &= A - B, \\
(-A) \cdot B &= -(A \cdot B) = A \cdot (-B), \\
-(-A) &= A, \\
A/B &= A \cdot (1/B), \\
\mu_{A\cdot(B+C)} &\leq \mu_{(A\cdot B)+(A\cdot C)}
\end{aligned}
$$

In the latter inequality, equality occurs if A crisp number ($A = x$).
The following situations may happper for fuzzy intervals:

$$
\begin{aligned}
A - A &\neq 0, \\
(A + B) - B &\neq A, \\
A/A &\neq 1, \\
(A/B) \cdot B &\neq A, \\
A \cdot (B + C) &\neq A \cdot B + A \cdot C.
\end{aligned}
$$