An MV-algebra for vagueness

Petr Hájek

Institute of Computer Sciences of the ASCR
Pod Vodárenskou věží 2, 182 08 Prague, Czech Republic
hajek@cs.cas.cz

The philosophers criticize sharp break between true cases (having truth value 1) and borderline cases (< 1). Here we offer a fuzzy semantics (a particular non-standard MV-algebra) where this is (somehow) handled. Łukasiewicz propositional and predicate logic with its standard, general and witnessed semantics is assumed to be known (and will be briefly recalled.)

Take the lexicographic product of two copies of the ordered additive group of reals (operations corordinatewise, ordering lexicographic); this is a linearly ordered Abelian group. Now our MV-chain K_{Re} is the interval $[(0,0),(1,0)]$ with the usual operations; thus a pair $(x,y) \in Re^2$ is in K_{Re} iff ($x = 0$ and $y \geq 0$) or ($0 < x < 1$ and y arbitrary) or ($x = 1$ and $y \leq 0$). For each $(x,y) \in K_{Re}$ let $\beta(x,y)$ be the set of all elements of K_{Re} with the first coordinate x (the neighborhood of (x,y) or the set of elements infinitely near to (x,y)). The set $\beta(1,0)$ is the set of degrees of almost full truth, say ft-degrees. Identify (0,0) with 0 and (1,0) with 1.

Recall that the operation $(a,b) * (c,d)$ is $\max[(0,0),(a+c-1,b+d)]$, in more details, $(a,b) * (c,d)$ is equal to

$(a+c-1, b+d)$ if $a+c > 1$,
$(0, \max(0, b+d))$ if $a+c = 1$,
$(0,0)$ if $a+c < 1$.

The set of all $(x,0) \in K_{Re}$ is (the domain of) a subalgebra of K_{Re} isomorphic with the standard MV-algebra $[0,1]_{\text{Ł}}$. The mapping $f(x,y) = x$ (projection to the first coordinate) is a homomorphism of K_{Re} to (the copy of) $[0,1]_{\text{Ł}}$. Form this it easily follows (for propositional calculus):

THEOREM 1 *The following sets are equal:*
the set $1Taut([0,1]_L)$ *of standard L-1-tautologies, the set* $1Taut(K_{Re})$ *of* K_{Re}*-1-tautologies,and the set* $\beta(1)Taut(K_{Re})$ *of* K_{Re}*-*$\beta(1)$*-tautologies. The same for satisfiability instead of tautologicity.*

Now turn to predicate calculus. The morphism f preserves all existing infinite sups and infs BUT be aware of what sups and infs in K_{Re} exist: a non-empty set $X \subseteq K_{Re}$ has infimum iff $f(X)$ has a minimum; similarly for sup and max (easy to see). Consequently,

LEMMA 2 *The homomorphic image* $f(M)$ *of a safe* K_{Re}*-structure is a witnessed* $[0,1]_L$*-structure. And each witnessed* $[0,1]_L$*-structure is a safe* K_{Re}*-structure modulo the embedding above.*

Now recall that the set of standard 1-tautologies of Łukasiewicz predicate calculus is equal to the set of standard witnessed 1-tautologies and similarly for satisfiables. (See my [2].) Our last lemma then gives:

THEOREM 3 *(1) A formula* φ *of predicate calculus is a* $\beta(1)$*-*K_{Re}*-tautology (i.e. has a value in* $\beta(1)$ *for each model over* K_{Re}*) iff it is a (witnessed) standard 1-tautology of Łukasiewicz predicate calculus Ł∀.*

(2) φ *is* $\beta(1)$*-*K_{Re}*-satisfiable (i.e. has a value form* $\beta(1)$ *in some model over* K_{Re}*) iff it is* (1)*-*K_{Re}*-satisfiable iff it is satisfiable in a (witnessed) standard model over Ł∀.*

COROLLARY 4 *The set* $\beta(1)TAUT\forall(K_{Re})$ *is* Π_2*-complete and the set* $\beta(1)SAT\forall(K_{Re}) = (1)SAT\forall(K_{Re})$ *is* Π_1*-complete.*

PROBLEM 5 *What is the arithmetical complexity of the set* $(1)TAUT\forall(K_{Re})$ *of 1-tautologies over* K_{Re}*? Is the set* $(1)TAUT\forall(K_{Re})$ *of 1-tautologies over* K_{Re} *equal to* $\beta(1)TAUT\forall(K_{Re})$*?*

These results will be compared with the results on Chang's MV-algebra and Komori's MV-algebras as presented in the paper [3].

References

[1] Hájek, P.: *Metamathematics of Fuzzy Logic*. Kluwer, 1998.

[2] Hájek, P.: On witnessed models in fuzzy logic. *Mathematical Logic Quarterly* 53 (2007), 66–77.

[3] Hájek, P. and Cintula, P.: Complexity issues in axiomatic extensions of Łukasiewicz logic. *Journal of Logic and Computation* 19 (2009), 245–260.