

Co-Stone Residuated Lattices

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Residuated lattices are the algebraic counterpart of monoidal logic; they include MTL-algebras, BL-algebras and MV-algebras. We recall that a *residuated lattice* is an algebraic structure $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$, with the first 4 operations binary and the last two constant, such that $(A, \vee, \wedge, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a commutative monoid and the following property, called *the law of residuation*, is satisfied: for all $a, b, c \in A$, $a \leq b \rightarrow c$ iff $a \odot b \leq c$, where \leq is the partial order of the lattice $(A, \vee, \wedge, 0, 1)$.

In [11] we gave an axiomatic purely algebraic definition of the reticulation of a residuated lattice, that we proved to be equivalent to the general notion of reticulation applied to residuated lattices, and which turned out to be very useful in practice. In the article that this abstract is based on, we present several applications for the reticulation, related to co-Stone algebras, applications in the form of transfers of properties between the category of bounded distributive lattices and the category of residuated lattices through the reticulation functor. This transfer of properties between different categories is the very purpose of the reticulation.

The co-Stone structures were introduced by us as being dual notions to Stone structures. Let A be a bounded distributive lattice or a residuated lattice; the definitions we are about to give are valid for both types of structures. For any non-empty subset X of A , the *co-annihilator of X* is the set $X^\top = \{a \in A \mid (\forall x \in X) a \vee x = 1\}$. In the case when X consists of a single element x , we denote the co-annihilator of X by x^\top and call it the *co-annihilator of x* . Also, we will denote $X^{\top\top} = (X^\top)^\top$ and $x^{\top\top} = (x^\top)^\top$. Let

us remark that, for A a bounded distributive lattice or a residuated lattice and for any $X \subseteq A$, X^\top is a filter of A . We will denote the Boolean center of a bounded distributive lattice or a residuated lattice A by $B(A)$.

Definition Let A be a bounded distributive lattice or a residuated lattice. Then A is said to be *co-Stone* (respectively *strongly co-Stone*) iff, for all $x \in A$ (respectively all $X \subseteq A$), there exists an element $e \in B(A)$ such that $x^\top = \langle e \rangle$ (respectively $X^\top = \langle e \rangle$).

Concerning co-Stone and strongly co-Stone structures (by structure we mean here bounded distributive lattice or residuated lattice), the first question that arises is whether they exist. Naturally, any strongly co-Stone structure is co-Stone and any complete co-Stone structure is strongly co-Stone. The answer to the question above is given by the fact that the trivial structure is strongly co-Stone and, moreover, any chain is strongly co-Stone, because a chain A clearly has all co-annihilators equal to $\{1\} = \langle 1 \rangle$, except for 1^\top , which is equal to $A = \langle 0 \rangle$.

We prove the fact that a residuated lattice is co-Stone iff its reticulation is co-Stone and the same is valid for strongly co-Stone structures, then we obtain a structure theorem for m -co-Stone residuated lattices, by transferring through the reticulation a known characterization of m -co-Stone bounded distributive lattices to residuated lattices. This is the first major example of a result that can be transferred through the reticulation functor from the category of bounded distributive lattices to the category of residuated lattices. It also permits us to state that a residuated lattice is m -co-Stone iff its reticulation is m -co-Stone. Here is the characterization of m -co-Stone residuated lattices that we are referring to:

Theorem Let m be an infinite cardinal. Then the following are equivalent:

- (I) for each subset X of A with $|X| \leq m$, there exists an element $e \in B(A)$ such that $X^\top = \langle e \rangle$;
- (II) A is a co-Stone residuated lattice and $B(A)$ is an m -complete Boolean algebra;
- (III) $A_{\top\top} = \{a^{\top\top} \mid a \in A\}$ is an m -complete Boolean sublattice of the lattice of filters of A ;

- (IV) for all $a, b \in A$, $(a \vee b)^\top = a^\top \vee b^\top$ and, for each subset X of A with $|X| \leq m$, there exists an element $x \in A$ such that $X^{\top\top} = x^\top$;
- (V) for each subset X of A with $|X| \leq m$, $X^\top \vee X^{\top\top} = A$.

We bring an argument for our choice of the definition of the co-Stone structures over another definition for them that can be found in mathematical literature, for instance in [5]: the fact that the notion with our definition is transferrable through the reticulation (while the alternate one is not and does not coincide with ours).

We then define the strongly co-Stone hull of a residuated lattice, in accordance with its definition for MV-algebras from [7] and for bounded distributive lattices from [6], show that it is preserved by the reticulation functor and exemplify its calculation for a finite residuated lattice. For proving that the reticulation functor for residuated lattices preserves the strongly co-Stone hull we are using the fact that it preserves inductive limits, that we proved in [13].

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