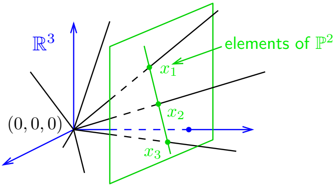
▶Homography



Projective plane $\mathbb{P}^2$: Vector space of dimension 3 excluding the zero vector, $\mathbb{R}^3 \setminus (0,0,0)$

but including 'points at infinity' and the 'line at infinity'

factorized to linear equivalence classes ('rays')

Collineation: Let x_1, x_2, x_3 be collinear points in $\mathbb{P}^2$ (coplanar rays in $\mathbb{R}^3$). Bijection $h\colon \mathbb{P}^2 \mapsto \mathbb{P}^2$ is a collineation iff $h(x_1)$, $h(x_2)$, $h(x_3)$ are collinear in $\mathbb{P}^2$ (coplanar in $\mathbb{R}^3$).

bijection = 1:1, onto

- collinear image points are mapped to collinear image points — lines are mapped to lines
- concurrent image lines are mapped to concurrent image lines — bijection!

  concurrent = intersecting at the same point
- point-line incidence is preserved

- mapping $h : \mathbb{P}^2 \to \mathbb{P}^2$ is a collineation iff there exists a non-singular 3×3 matrix $\mathbf{H}$ s.t.

$$h(x) \simeq \mathbf{H}\,\underline{\mathbf{x}} \quad \text{for all } \underline{\mathbf{x}} \in \mathbb{P}^2$$

- homogeneous matrix representant: $\det \mathbf{H} = 1$
- in this course we will use the term **homography** but mean collineation

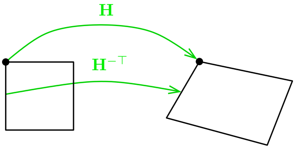
Some Homographic Tasters

Rectification of camera rotation: Slides 63 (geometry), 120 (homography estimation)

Homographic Mouse for Visual Odometry: Slide TBD

illustrations courtesy of AMSL Racing Team, Meiji University and LIBVISO: Library for VISual Odometry

▶Mapping Points and Lines by Homography



$\underline{\mathbf{m}}' \simeq \mathbf{H}\,\underline{\mathbf{m}}$	image point	
$\underline{\mathbf{n}}' \simeq \mathbf{H}^{-\top}\underline{\mathbf{n}}$	image line	$\mathbf{H}^{-\top} = (\mathbf{H}^{-1})^{\top} = (\mathbf{H}^{\top})^{-1}$

- incidence is preserved: $(\underline{\mathbf{m}}')^{\top}\underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^{\top}\mathbf{H}^{\top}\mathbf{H}^{-\top}\underline{\mathbf{n}} = \underline{\mathbf{m}}^{\top}\underline{\mathbf{n}} = 0$

1. collineation has 8 DOF; it is given by 4 correspondences (points, lines) in a general position
2. extending pixel coordinates to homogeneous coordinates $\underline{\mathbf{m}} = (u, v, \mathbf{1})$
3. mapping by homography, eg. $\underline{\mathbf{m}}' = \mathbf{H}\,\underline{\mathbf{m}}$
4. conversion of the result $\underline{\mathbf{m}}' = (m_1', m_2', m_3')$ to canonical coordinates (pixels):

$$u' = \frac{m_1'}{m_3'}\mathbf{1}, \qquad v' = \frac{m_2'}{m_3'}\mathbf{1}$$

5. can use the unity for the homogeneous coordinate on one side of the equation only!

Elementary Decomposition of a Homography

Unique decompositions: $\mathbf{H} = \mathbf{H}_S\,\mathbf{H}_A\,\mathbf{H}_P \quad (= \mathbf{H}'_P\,\mathbf{H}'_A\,\mathbf{H}'_S)$

$$\mathbf{H}_S = \begin{bmatrix} s\,\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \qquad \text{similarity}$$

$$\mathbf{H}_A = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \qquad \text{special affine}$$

$$\mathbf{H}_P = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\top & w \end{bmatrix} \qquad \text{special projective}$$

$\mathbf{K}$ – upper triangular matrix with positive diagonal entries
$\mathbf{R}$ – orthogonal, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det\mathbf{R} = 1$
$s, w \in \mathbb{R}$, $s > 0$, $w \neq 0$

$$\mathbf{H} = \begin{bmatrix} s\mathbf{R}\mathbf{K} + \mathbf{t}\,\mathbf{v}^\top & w\,\mathbf{t} \\ \mathbf{v}^\top & w \end{bmatrix}$$

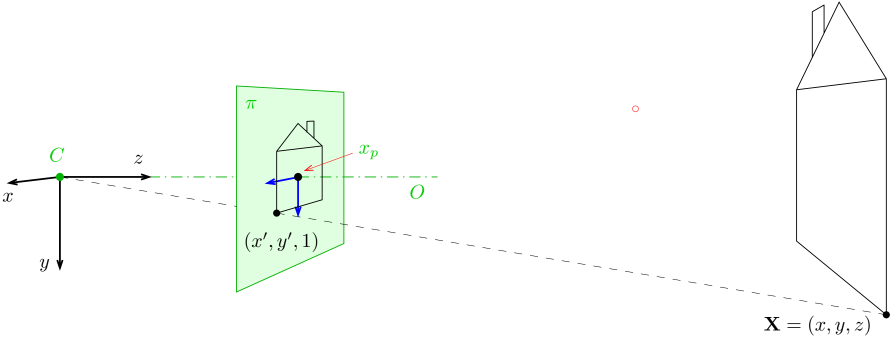
- must use 'skinny' QR decomposition, which is unique [Golub & van Loan 1996, Sec. 5.2.6]
- $\mathbf{H}_S$, $\mathbf{H}_A$, $\mathbf{H}_P$ are collineation subgroups
  (eg. $\mathbf{K} = \mathbf{K}_1\mathbf{K}_2$, $\mathbf{K}^{-1}$, $\mathbf{I}$ are all upper triangular with unit determinant, associativity holds)

Homography Subgroups

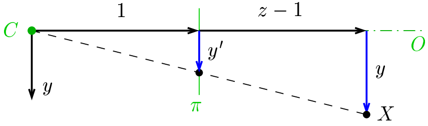
group	DOF	matrix	invariant properties
projective	8	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$	incidence, concurrency, colinearity, cross-ratio, order of contact (intersection, tangency, inflection), tangent discontinuities and cusps.
affine	6	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$	<u>all above plus:</u> parallelism, ratio of areas, ratio of lengths on parallel lines, linear combinations of vectors (e.g. midpoints), line at infinity $\underline{\mathbf{n}}_\infty$ (not pointwise)
similarity	4	$\begin{bmatrix} s\,\cos\phi & s\,\sin\phi & t_x \\ -s\,\sin\phi & s\,\cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	<u>all above plus:</u> ratio of lengths, angle, the circular points $I = (1, i, 0)$, $J = (1, -i, 0)$.
Euclidean	3	$\begin{bmatrix} \cos\phi & \sin\phi & t_x \\ -\sin\phi & \cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	<u>all above plus:</u> length, area

[H.8.7, P.2, p. 44]

►Canonical Perspective Camera (Pinhole Camera, Camera Obscura)



1. right-handed canonical coordinate system (x, y, z)
2. origin = center of projection C
3. image plane π at <u>unit</u> distance from C
4. optical axis O is perpendicular to π
5. principal point x_p: intersection of O and π
6. in this picture we are looking 'down the street'
7. perspective camera is given by C and π



projected point in the natural image coordinate system:

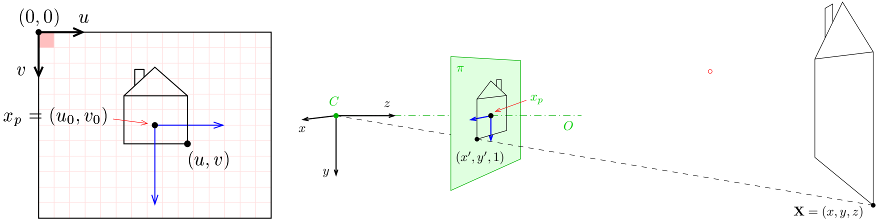
$$\frac{y'}{1} = y' = \frac{y}{1 + z - 1} = \frac{y}{z}, \qquad x' = \frac{x}{z}$$

▶Natural and Canonical Image Coordinate Systems

projected point **in canonical camera**

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix}^\top = \begin{bmatrix} \frac{x}{z}, & \frac{y}{z}, & 1 \end{bmatrix}^\top = \frac{1}{z}\begin{bmatrix} x, & y, & z \end{bmatrix}^\top \simeq \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}_0} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{P}_0\, \underline{\mathbf{X}}$$

projected point **in scanned image** notice the chimney!



$$u = f\,\frac{x}{z} + u_0$$
$$v = f\,\frac{y}{z} + v_0$$

$$\frac{1}{z}\begin{bmatrix} f\,x + z\,u_0 \\ f\,y + z\,v_0 \\ z \end{bmatrix} \simeq \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0\, \underline{\mathbf{X}} = \mathbf{P}\, \underline{\mathbf{X}}$$

- ‘calibration’ matrix $\mathbf{K}$ transforms canonical camera $\mathbf{P}_0$ to standard projective camera $\mathbf{P}$

►Computing with Perspective Camera Projection Matrix

$$\underline{\mathbf{m}} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \simeq \begin{bmatrix} x + \frac{z}{f} u_0 \\ y + \frac{z}{f} v_0 \\ \frac{z}{f} \end{bmatrix}$$

$$\frac{m_1}{m_3} = \frac{f\,x}{z} + u_0 = u, \qquad \frac{m_2}{m_3} = \frac{f\,y}{z} + v_0 = v \quad \text{when} \quad m_3 \neq 0$$

f – 'focal length' – converts length ratios to pixels, $[f] = \mathrm{px}$, $f > 0$

(u_0, v_0) – principal point in pixels

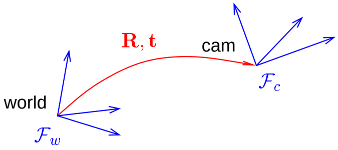
Perspective Camera:

1. dimension reduction — since $\mathbf{P} \in \mathbb{R}^{3,4}$
2. nonlinear unit change $\mathbf{1} \mapsto \mathbf{1} \cdot z/f$ — since $\underline{\mathbf{m}} \simeq (x, y, z/f)$

   for convenience we use $P_{11} = P_{22} = f$ rather than $P_{33} = 1/f$ and the u_0, v_0 in relative units
3. $m_3 = 0$ represents points at infinity in image plane π $(z = 0)$

▶Changing The Outer (World) Reference Frame

A transformation of a point from the world to camera coordinate system:



$$\mathbf{X}_c = \mathbf{R}\,\mathbf{X}_w + \mathbf{t}$$

$\mathbf{R}$ – camera rotation matrix — world orientation in the camera coordinate frame

$\mathbf{t}$ – camera translation vector — world origin in the camera coordinate frame

$$\mathbf{P}\,\underline{\mathbf{X}}_c = \mathbf{K}\mathbf{P}_0 \begin{bmatrix} \mathbf{X}_c \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0 \begin{bmatrix} \mathbf{R}\mathbf{X}_w + \mathbf{t} \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0 \underbrace{\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} \mathbf{X}_w \\ 1 \end{bmatrix} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \underline{\mathbf{X}}_w$$

$\mathbf{P}_0$ selects the first 3 rows of $\mathbf{T}$ and discards the last row

- $\mathbf{R}$ is rotation, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det\mathbf{R} = +1$ — $\mathbf{I} \in \mathbb{R}^{3,3}$ identity matrix
- 6 **extrinsic parameters**: 3 rotation angles (Euler theorem), 3 translation components
- alternative, often used, camera representations

$$\mathbf{P} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix}$$

  $\mathbf{C}$ – camera position in the world reference frame — $\mathbf{t} = -\mathbf{R}\mathbf{C}$

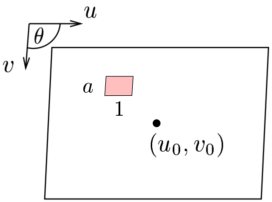
  $\mathbf{r}_3^\top$ – optical axis in the world reference frame — third row of $\mathbf{R}$: $\mathbf{r}_3 = \mathbf{R}^{-1}[0, 0, 1]^\top$

- we can save some conversion and computation by noting that $\mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix}\underline{\mathbf{X}} = \mathbf{K}\mathbf{R}(\mathbf{X} - \mathbf{C})$

▶Changing the Inner (Image) Reference Frame

The general form of calibration matrix $\mathbf{K}$ includes

- digitization raster skew angle θ
- pixel aspect ratio a



$$\mathbf{K} = \begin{bmatrix} f & -f\cot\theta & u_0 \\ 0 & f/(a\sin\theta) & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

units: $[f] = \mathrm{px}$, $[u_0] = \mathrm{px}$, $[v_0] = \mathrm{px}$, $[a] = 1$

⊛ H1; 2pt: Verify this $\mathbf{K}$; hints: $u'\mathbf{e}_{u'} + v'\mathbf{e}_{v'} = u\mathbf{e}_u + v\mathbf{e}_v$, boldface are basis vectors, $\mathbf{K}$ maps from an orthogonal system to a skewed system $[w'u', w'v', w']^\top = \mathbf{K}[u, v, 1]^\top$; first skew then sampling then shift by u_0, v_0 deadline LD+2 wk

general finite perspective camera has 11 parameters:

- 5 intrinsic parameters: f, u_0, v_0, a, θ
- 6 extrinsic parameters: $\mathbf{t}$, $\mathbf{R}(\alpha, \beta, \gamma)$

finite camera: $\det \mathbf{K} \neq 0$

$$\underline{\mathbf{m}} \simeq \mathbf{P}\underline{\mathbf{X}}, \qquad \mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

a recipe for filling $\mathbf{P}$

Representation Theorem: The set of projection matrices $\mathbf{P}$ of finite projective cameras is isomorphic to the set of homogeneous 3×4 matrices with the left hand 3×3 submatrix $\mathbf{Q}$ non-singular.

▶Projection Matrix Decomposition

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \quad \longrightarrow \quad \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

$\mathbf{Q} \in \mathbb{R}^{3,3}$	full rank (if finite perspective cam.)
$\mathbf{K} \in \mathbb{R}^{3,3}$	upper triangular with positive diagonal entries
$\mathbf{R} \in \mathbb{R}^{3,3}$	rotation: $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = +1$

1. $\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}$ see next
2. RQ decomposition of $\mathbf{Q} = \mathbf{K}\mathbf{R}$ using three Givens rotations [H&Z, p. 579]
$$\mathbf{K} = \mathbf{Q}\,\underbrace{\mathbf{R}_{32}\mathbf{R}_{31}\mathbf{R}_{21}}_{\mathbf{R}^{-1}}$$
3. $\mathbf{t} = -\mathbf{R}\mathbf{C}$

$\mathbf{R}_{ij}$ zeroes element ij in $\mathbf{Q}$ affecting only columns i and j and the sequence preserves previously zeroed elements, e.g.

$$\mathbf{R}_{32} = \begin{bmatrix}1 & 0 & 0\\ 0 & c & s\\ 0 & -s & c\end{bmatrix}, \quad c^2 + s^2 = 1, \quad \text{gives} \quad c = \frac{q_{33}}{\sqrt{q_{32}^2 + q_{33}^2}} \quad s = \frac{q_{32}}{\sqrt{q_{32}^2 + q_{33}^2}}$$

⊛ P1; 1pt: Multiply known matrices $\mathbf{K}$, $\mathbf{R}$ and then decompose back; discuss numerical errors

- RQ decomposition nonuniqueness: $\mathbf{K}\mathbf{R} = \mathbf{K}\mathbf{T}^{-1}\mathbf{T}\mathbf{R}$, where $\mathbf{T} = \mathrm{diag}(-1, -1, 1)$ is also a rotation, we must correct the result so that the diagonal elements of $\mathbf{K}$ are all positive
  'skinny' RQ decomposition
- care must be taken to avoid overflow, see [Golub & van Loan 1996, sec. 5.2]

RQ Decomposition Step

```
Q = Array[q, {3, 3}];
R32 = {{1, 0, 0}, {0, c, s}, {0, -s, c}};
R32 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}$$

```
Q1 = Q.R32;
Q1 // MatrixForm
s1 = Solve[{Q1[[3]][[2]] == 0, c^2 + s^2 == 1}, {c, s}];
s1 = s1[[2]]
Q1 /. s1 // Simplify // MatrixForm
```

$$\begin{pmatrix} q[1,1] & c\,q[1,2] - s\,q[1,3] & s\,q[1,2] + c\,q[1,3] \\ q[2,1] & c\,q[2,2] - s\,q[2,3] & s\,q[2,2] + c\,q[2,3] \\ q[3,1] & c\,q[3,2] - s\,q[3,3] & s\,q[3,2] + c\,q[3,3] \end{pmatrix}$$

$$\left\{c \to \frac{q[3,3]}{\sqrt{q[3,2]^2 + q[3,3]^2}},\ s \to \frac{q[3,2]}{\sqrt{q[3,2]^2 + q[3,3]^2}}\right\}$$

$$\begin{pmatrix} q[1,1] & \frac{-q[1,3]\,q[3,2]+q[1,2]\,q[3,3]}{\sqrt{q[3,2]^2+q[3,3]^2}} & \frac{q[1,2]\,q[3,2]+q[1,3]\,q[3,3]}{\sqrt{q[3,2]^2+q[3,3]^2}} \\ q[2,1] & \frac{-q[2,3]\,q[3,2]+q[2,2]\,q[3,3]}{\sqrt{q[3,2]^2+q[3,3]^2}} & \frac{q[2,2]\,q[3,2]+q[2,3]\,q[3,3]}{\sqrt{q[3,2]^2+q[3,3]^2}} \\ q[3,1] & 0 & \sqrt{q[3,2]^2 + q[3,3]^2} \end{pmatrix}$$

▶Center of Projection

Observation: finite $\mathbf{P}$ has a non-trivial right null-space rank 3 but 4 columns

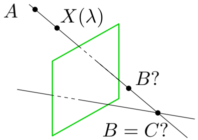
Theorem

Let there be $\underline{\mathbf{B}} \neq \mathbf{0}$ s.t. $\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{0}$. Then $\underline{\mathbf{B}}$ is equal to the projection center $\underline{\mathbf{C}}$ (in world coordinate frame).

Proof.

1. Consider spatial line $A\,B$ (B is given). We can write

$$\underline{\mathbf{X}}(\lambda) \simeq \underline{\mathbf{A}} + \lambda\,\underline{\mathbf{B}}, \qquad \lambda \in \mathbb{R}$$



2. it images to

$$\mathbf{P}\underline{\mathbf{X}}(\lambda) \simeq \mathbf{P}\,\underline{\mathbf{A}} + \lambda\,\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{P}\,\underline{\mathbf{A}}$$

- the whole line images to a single point $\Rightarrow$ it must pass through the optical center of $\mathbf{P}$
- this holds for all choices of A $\Rightarrow$ the only common point of the lines is the C, i.e. $\underline{\mathbf{B}} \simeq \underline{\mathbf{C}}$ □

Hence

$$\mathbf{0} = \mathbf{P}\,\underline{\mathbf{C}} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}\begin{bmatrix}\mathbf{C}\\ 1\end{bmatrix} = \mathbf{Q}\,\mathbf{C} + \mathbf{q} \Rightarrow \boxed{\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}}$$

$\underline{\mathbf{C}} = (c_j)$, where $c_j = (-1)^j \det \mathbf{P}^{(j)}$, in which $\mathbf{P}^{(j)}$ is $\mathbf{P}$ with column j dropped

Matlab: `C_homo = null(P);` or `C = -Q\q;`