Geometry of image formation

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Talk Outline

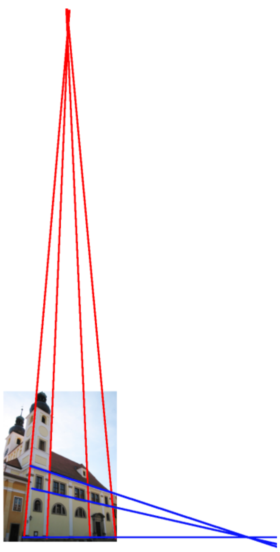
- Pinhole model
- Camera parameters
- Estimation of the parameters—Camera calibration

Motivation

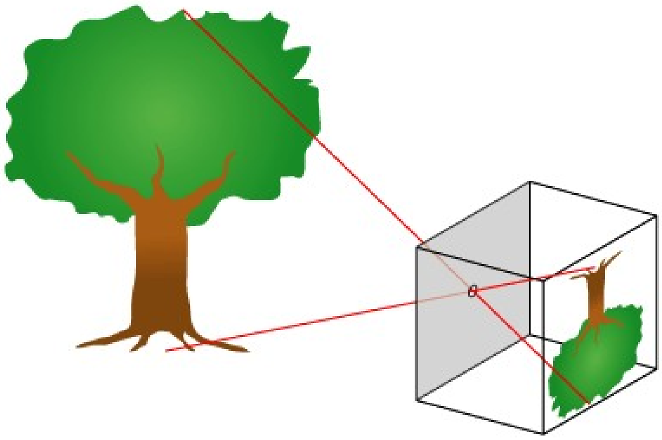
- parallel lines
- window sizes
- image units
- distance from the camera

What will we learn

- how does the 3D world project to 2D image plane?
- how is a camera modeled?
- how can we estimate the camera model?

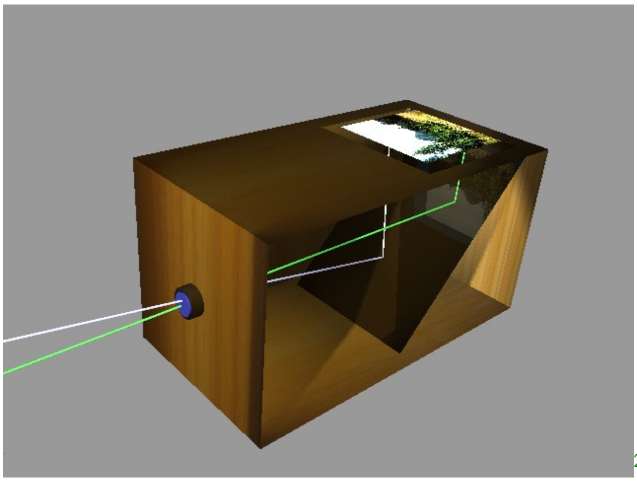


Pinhole camera


[1]

[1] http://en.wikipedia.org/wiki/Pinhole_camera

Camera Obscura


[2]

[2] http://en.wikipedia.org/wiki/Camera_obscura

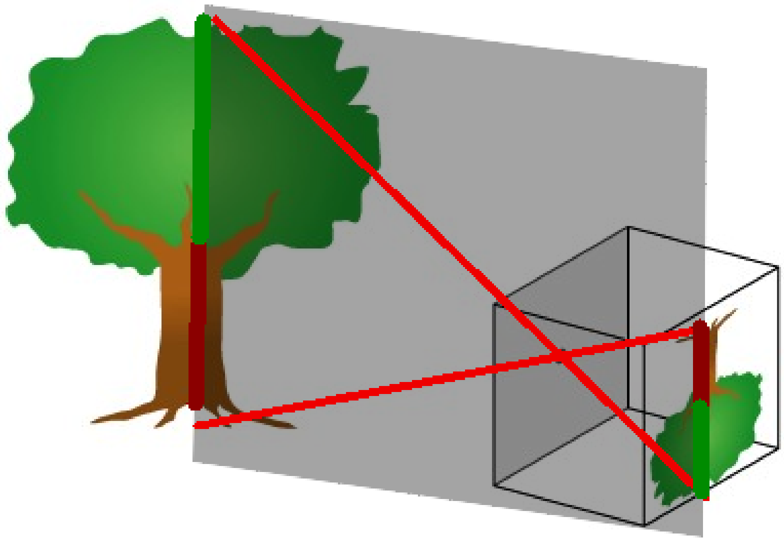
Camera Obscura — room-sized


[3]

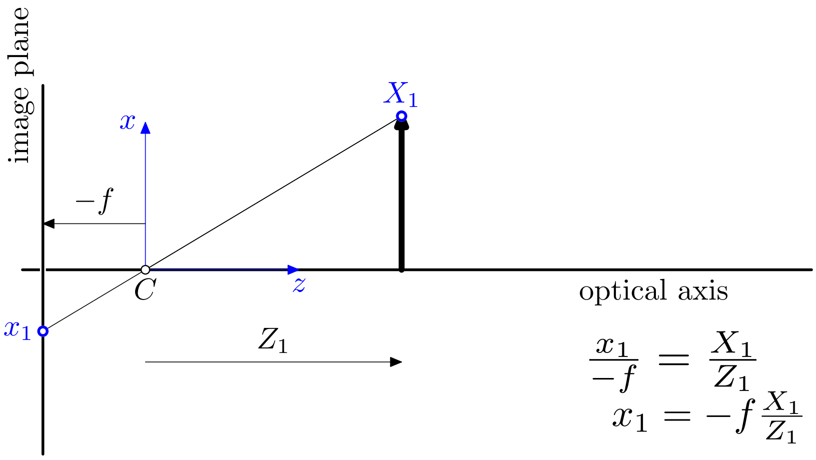
Used by the art department at the UNC at Chapel Hill

[3] http://en.wikipedia.org/wiki/Camera_obscura

1D Pinhole camera

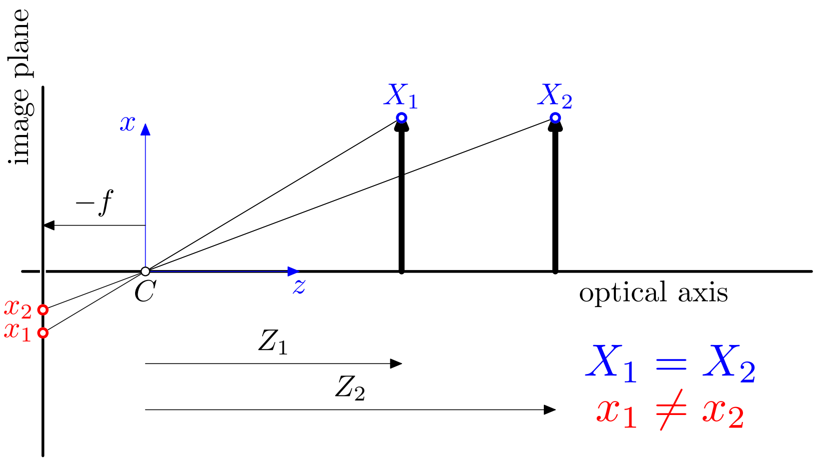


1D Pinhole camera projects 2D to 1D



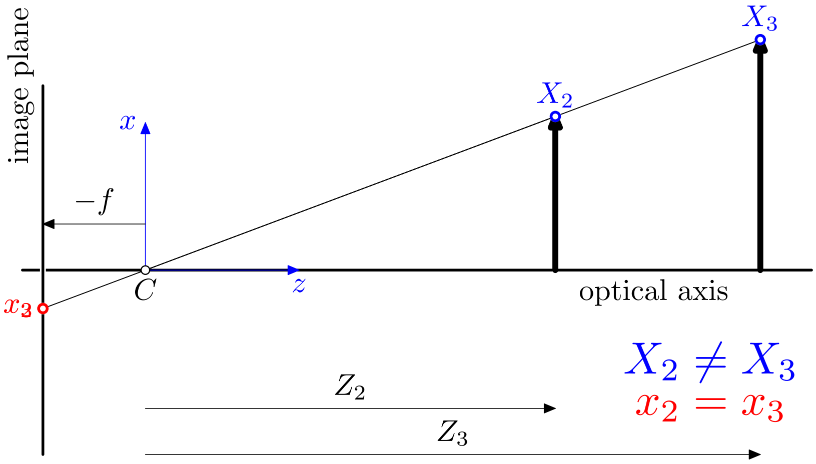
$$\frac{x_1}{-f} = \frac{X_1}{Z_1}$$
$$x_1 = -f\,\frac{X_1}{Z_1}$$

Problems with perspective I

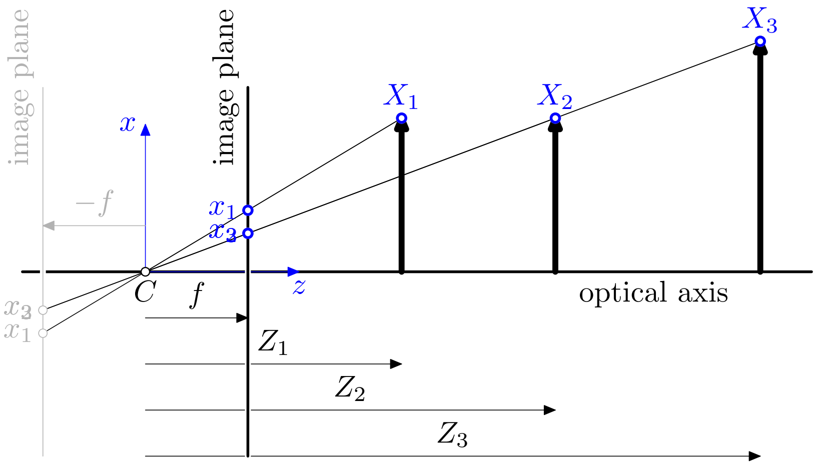


$$X_1 = X_2$$
$$x_1 \neq x_2$$

Problems with perspective II

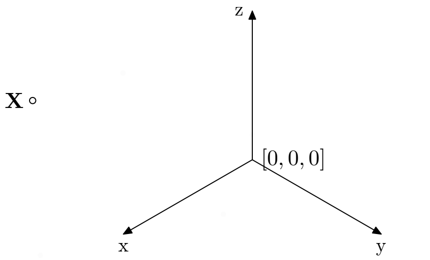


Get rid of the (−) sign

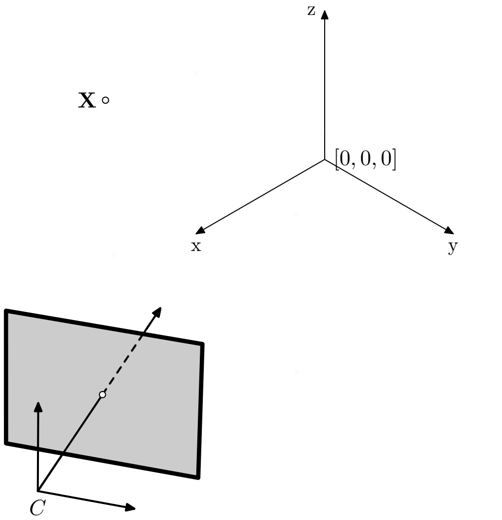


How does the 3D world
project to the 2D image plane?

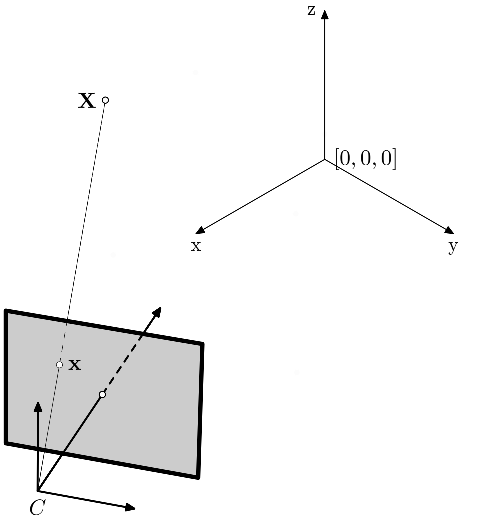
A 3D point $\mathbf{X}$ in a world coordinate system



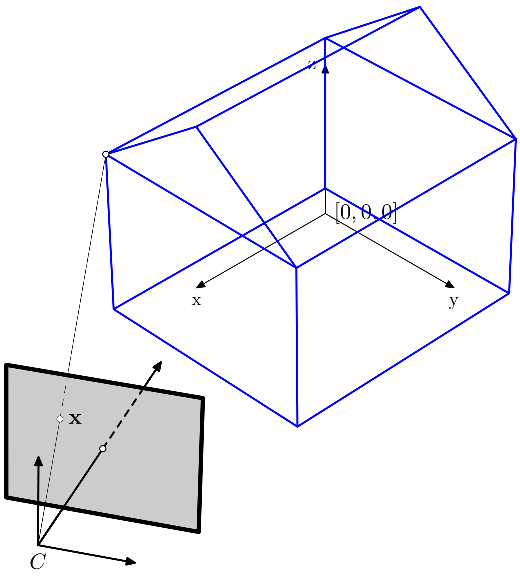
A pinhole camera observes a scene



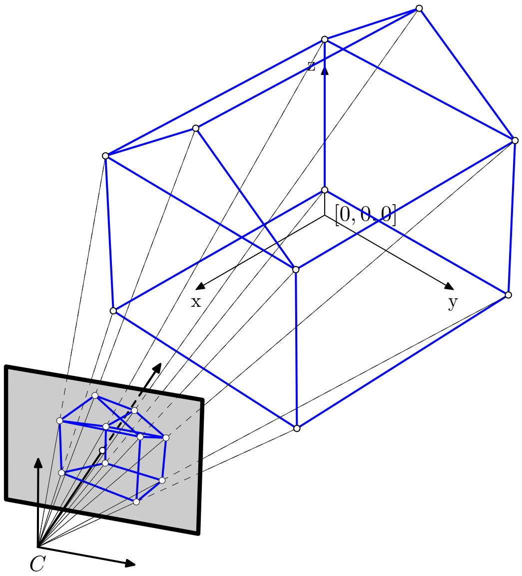
Point $\mathbf{X}$ projects to the image plane, point $\mathbf{x}$



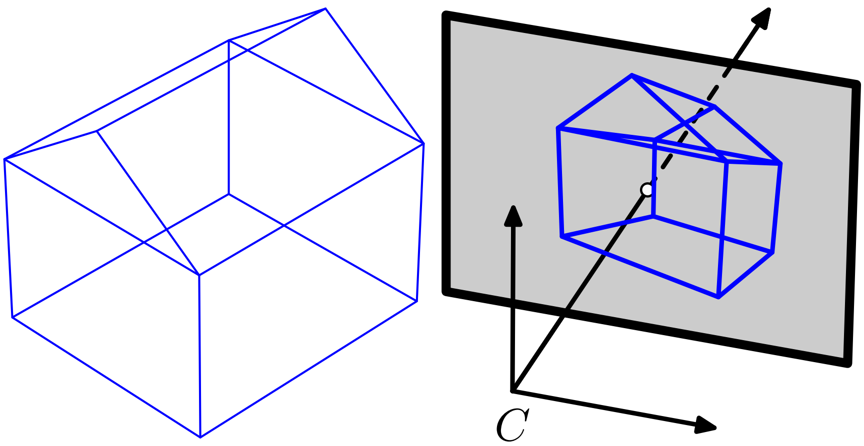
Scene projection



Scene projection



3D Scene projection – observations



- 3D lines project to 2D lines
- but the angles change, parallel lines are no more parallel.
- area ratios change, note the front and backside of the house

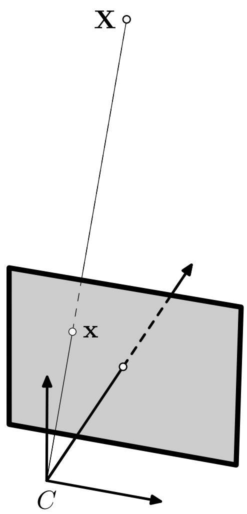
Put the sketches into equations

3D $\longrightarrow$ 2D Projection

We remember that: $\mathbf{x} = [\frac{fX}{Z}, \frac{fY}{Z}]^\top$

$$\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} \simeq \begin{bmatrix} fX \\ fY \\ Z \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} \simeq \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$



Use the homegeneous coordinates[4]

$$\lambda_{[1\times1]}\mathbf{x}_{[3\times1]} = \mathtt{K}_{[3\times3]} \left[\ \mathtt{I}|\mathbf{0}\ \right] \mathbf{X}_{[4\times1]}$$

but . . .

[4] for the notation conventions, see the talk notes

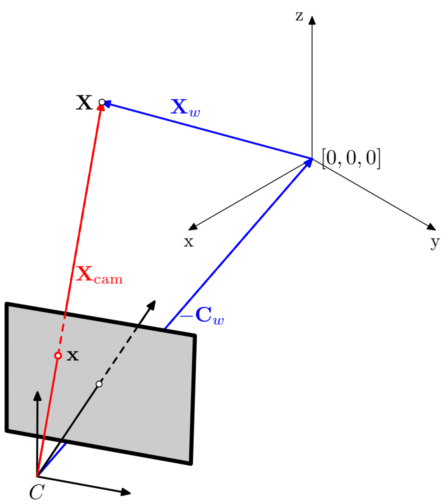
. . . we need the $\mathbf{X}$ in camera coordinate system

Rotate the vector:

$$\mathbf{X} = \mathtt{R}(\mathbf{X}_w - \mathbf{C}_w)$$

$\mathtt{R}$ is a 3×3 rotation matrix. The point coordinates $\mathbf{X}$ are now in the camera frame.
Use homogeneous coordinates to get a matrix equation

$$\begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{R} & -\mathtt{R}\mathbf{C}_w \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_w \\ 1 \end{bmatrix}$$



The camera center $\mathbf{C}_w$ is often replaced by the translation vector

$$\mathbf{t} = -\mathtt{R}\mathbf{C}_w$$

External (extrinsic parameters)

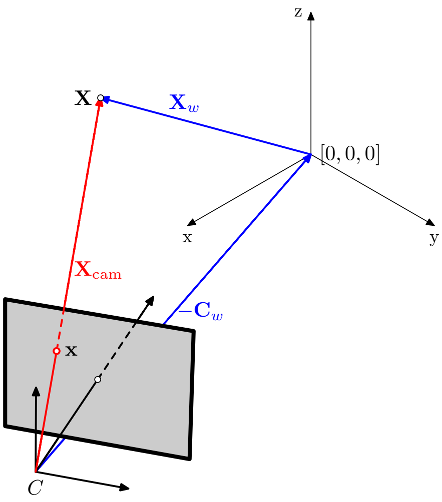
The translation vector $\mathbf{t}$ and the rotation matrix $\mathtt{R}$ are called External parameters of the camera.

$$\mathbf{x} \simeq \mathtt{K}\,[\,\mathtt{I}|\mathbf{0}\,]\begin{bmatrix}\mathbf{X}\\1\end{bmatrix}$$

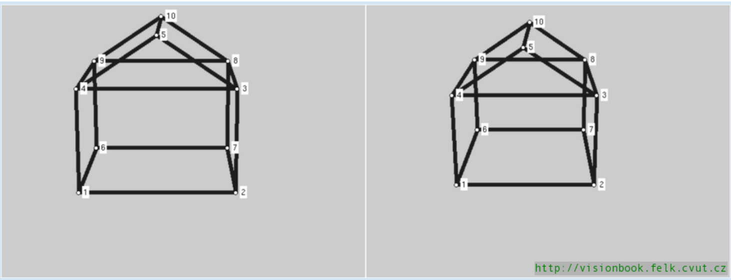
$$\lambda\mathbf{x} = \mathtt{K}\,[\,\mathtt{R}\;\;\mathbf{t}\,]\begin{bmatrix}\mathbf{X}_w\\1\end{bmatrix}$$

Camera parameters (so far): $f, \mathtt{R}, \mathbf{t}$

Is it all? What can we model?



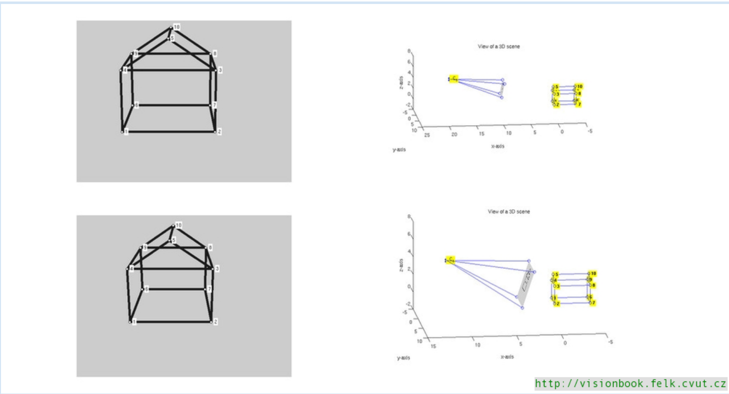
What is the geometry good for?



video: Zoom out vs. motion away from scene

- How would you characterize the difference?
- Would you guess the motion type?

What is the geometry good for?

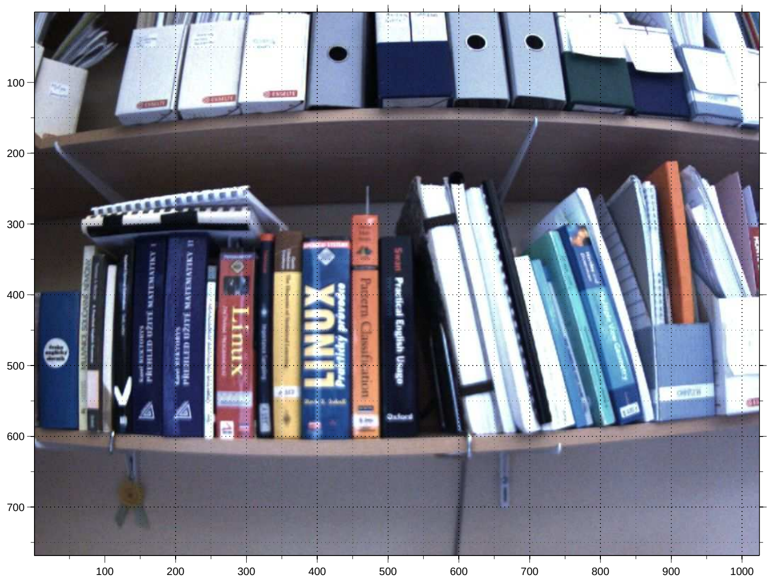


video: Zoom out vs. motion away from scene

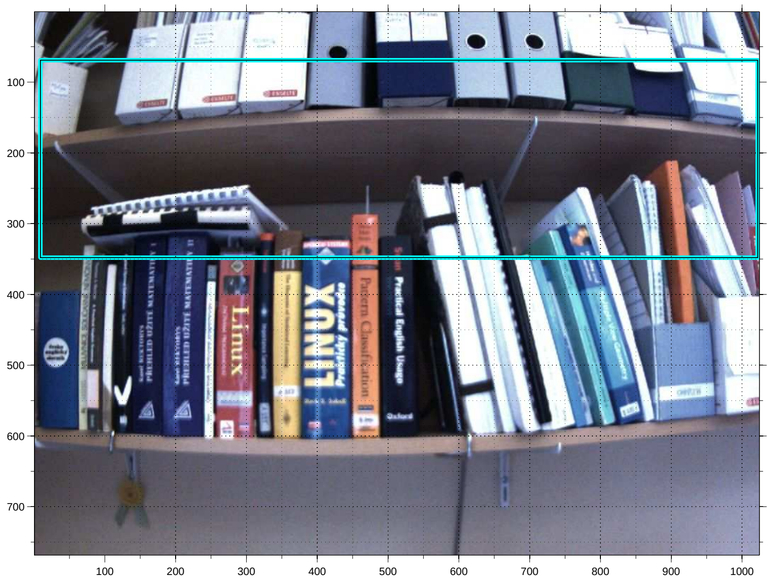
Enough geometry[5], look at real images

[5]just for a moment

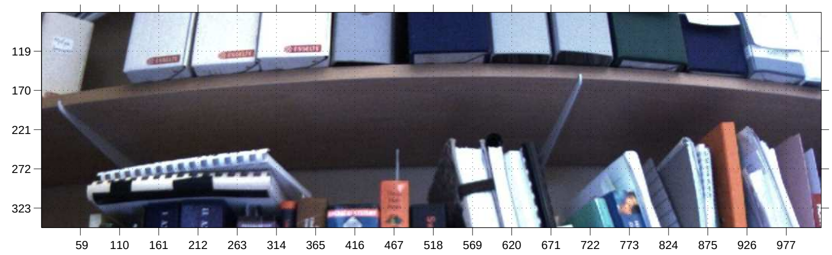
From geometry to pixels and back again



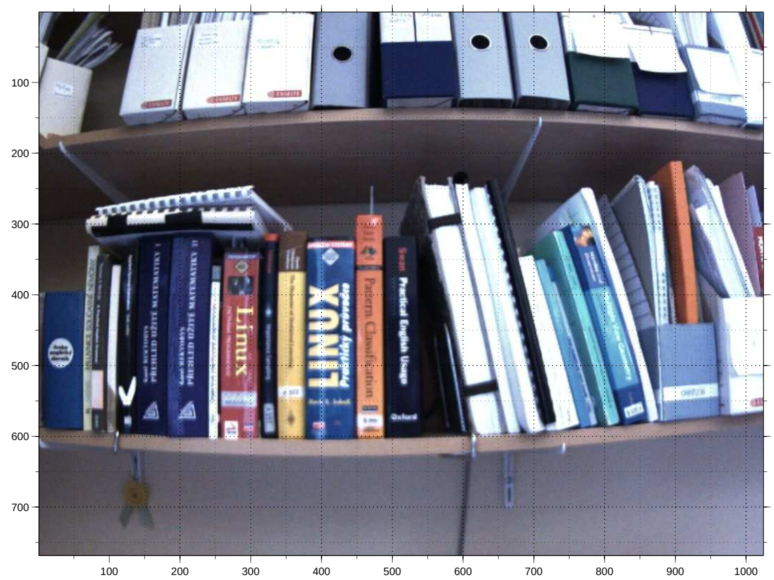
Problems with pixels



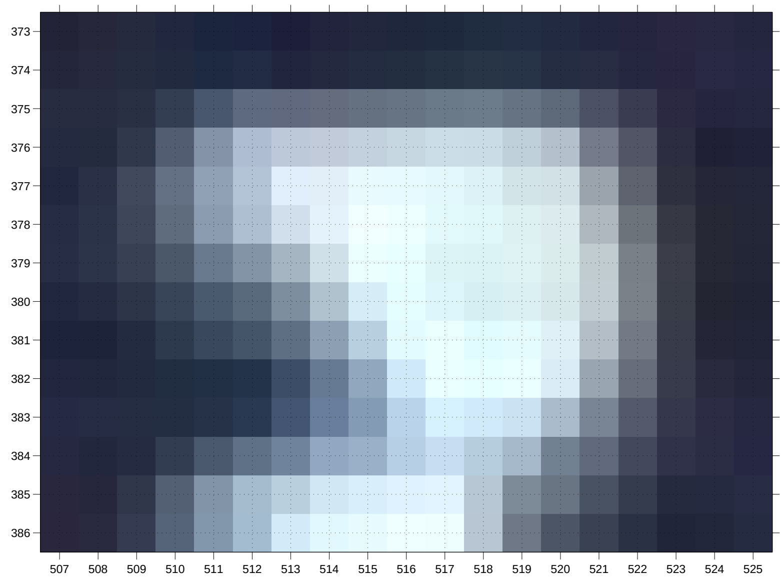
Is this a stright line?



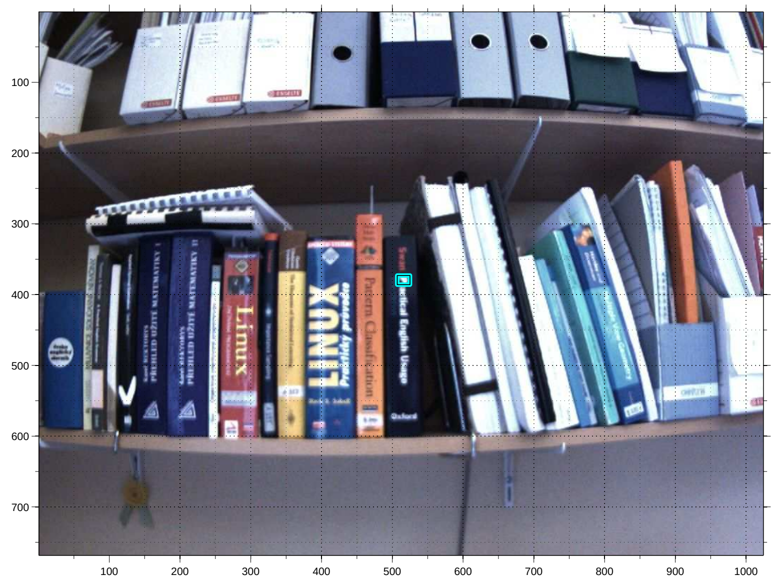
Problems with pixels



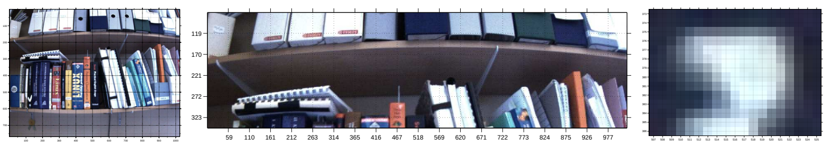
What are we looking at?



Did you recognize it?



Pixel images revisited



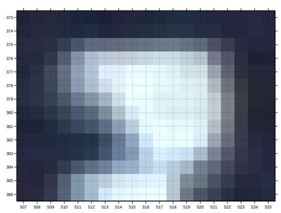
- There are no negative coordinates. Where is the principal point?
- Lines are not lines any more.
- Pixels, considered independently, do not carry much information.

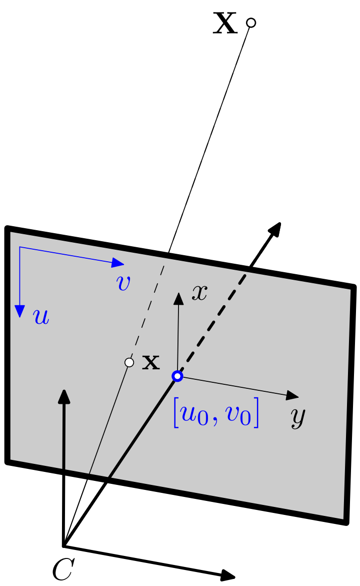
Pixel coordinate system

Assume normalized geometrical coordinates $\mathbf{x} = [x, y, 1]^\top$

$$
\begin{aligned}
u &= m_u(-x) + u_0 \\
v &= m_v y + v_0
\end{aligned}
$$

where m_u, m_v are sizes of the pixels and $[u_0, v_0]^\top$ are coordinates of the principal point.





Put pixels and geometry together

From 3D to image coordinates: $\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathtt{R} & \mathbf{t} \end{bmatrix} \mathbf{X}_{[4\times 1]}$

From normalized coordinates to pixels: $\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} -m_u & 0 & u_0 \\ 0 & m_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Put them together: $\frac{1}{\lambda} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} -fm_u & 0 & u_0 \\ 0 & fm_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathtt{R} & \mathbf{t} \end{bmatrix} \mathbf{X}$

Finally: $\mathbf{u} \simeq \mathtt{K} \begin{bmatrix} \mathtt{R} & \mathbf{t} \end{bmatrix} \mathbf{X}$

Introducing a 3×4 camera projection matrix $\mathtt{P}$: $\boxed{\mathbf{u} \simeq \mathtt{P}\mathbf{X}}$

Non-linear distortion

Several models exist. Less standardized than the linear model. We will consider a simple on-parameter radial distortion. $\mathbf{x}_n$ denote the linear image coordinates, $\mathbf{x}_d$ the distorted ones.

$$\mathbf{x}_d = (1 + \kappa r^2)\mathbf{x}_n$$

where κ is the distortion parameter, and $r^2 = x_n^2 + y_n^2$ is the distance from the principal point.

Observable are the distorted pixel coordinates

$$\mathbf{u}_d = \mathtt{K}\mathbf{x}_d$$

Assume that we know κ. How to get the lines back?

Undoing Radial Distortion

From pixels to distorted image coordinates: $\mathbf{x}_d = \mathtt{K}^{-1}\mathbf{u}_d$

From distorted to linear image coordinates: $\mathbf{x}_n = \frac{\mathbf{x}_\mathrm{d}}{1+\kappa r^2}$

Where is the problem? $r^2 = x_n^2 + y_n^2$. We have unknowns on both sides of the equation.

Iterative solution:

1. initialize $\mathbf{x}_n = \mathbf{x}_d$
2. $r^2 = x_n^2 + y_n^2$
3. compute $\mathbf{x}_n = \frac{\mathbf{x}_\mathrm{d}}{1+\kappa r^2}$
4. go to 2. (and repeat few times)

And back to pixels $\mathbf{u}_n = \mathtt{K}\mathbf{x}_n$

Undoing Radial Distortion

video

Estimation of camera parameters—camera calibration

The goal: estimate the 3×4 camera projection matrix $\mathtt{P}$ and possibly the parameters of the non-linear distortion κ from images.

Assume a known projection $[u, v]^\top$ of a 3D point $\mathbf{X}$ with known coordinates

$$\begin{bmatrix} \lambda u \\ \lambda v \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{P}_1^\top \\ \mathbf{P}_2^\top \\ \mathbf{P}_3^\top \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\frac{\lambda u}{\lambda} = \frac{\mathbf{P}_1^\top \mathbf{X}}{\mathbf{P}_3^\top \mathbf{X}} \quad \text{and} \quad \frac{\lambda v}{\lambda} = \frac{\mathbf{P}_2^\top \mathbf{X}}{\mathbf{P}_3^\top \mathbf{X}}$$

Re-arrange and assume[6] $\lambda \neq 0$ to get set of homegeneous equations

$$\begin{aligned} u\mathbf{X}^\top \mathbf{P}_3 - \mathbf{X}^\top \mathbf{P}_1 &= 0 \\ v\mathbf{X}^\top \mathbf{P}_3 - \mathbf{X}^\top \mathbf{P}_2 &= 0 \end{aligned}$$

[6] see some notes about $\lambda = 0$ in the talk notes

Estimation of the $\mathtt{P}$ matrix

$$\begin{aligned} u\mathbf{X}^\top \mathbf{P}_3 - \mathbf{X}^\top \mathbf{P}_1 &= 0 \\ v\mathbf{X}^\top \mathbf{P}_3 - \mathbf{X}^\top \mathbf{P}_2 &= 0 \end{aligned}$$

Re-shuffle into a matrix form:

$$\underbrace{\begin{bmatrix} -\mathbf{X}^\top & \mathbf{0}^\top & u\mathbf{X}^\top \\ \mathbf{0}^\top & -\mathbf{X}^\top & v\mathbf{X}^\top \end{bmatrix}}_{\mathtt{A}_{[2\times 12]}} \underbrace{\begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}}_{\mathbf{p}_{[12\times 1]}} = \mathbf{0}_{[2\times 1]}$$

A correspondece $\mathbf{u}_i \leftrightarrow \mathbf{X}_i$ forms two homogeneous equations. $\mathtt{P}$ has 12 parameters but scale does not matter. We need at least 6 2D $\leftrightarrow$ 3D pairs to get a solution. We constitute $\mathtt{A}_{[\geq 12\times 12]}$ data matrix and solve

$$\mathbf{p}^* = \operatorname{argmin} \|\mathtt{A}\mathbf{p}\| \quad \text{subject to} \quad \|\mathbf{p}\| = 1$$

which is a constrained LSQ problem. $\mathbf{p}^*$ minimizes algebraic error

Decomposition of P into the calibration parameters

$$\mathtt{P} = \begin{bmatrix} \mathtt{KR} & \mathtt{K}\mathbf{t} \end{bmatrix} \quad \text{and} \quad \mathbf{C} = -\mathtt{R}^{-1}\mathbf{t}$$

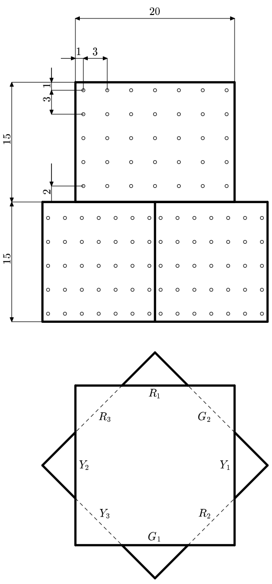
We know that $\mathtt{R}$ should be 3×3 orthonormal, and $\mathtt{K}$ upper triangular.

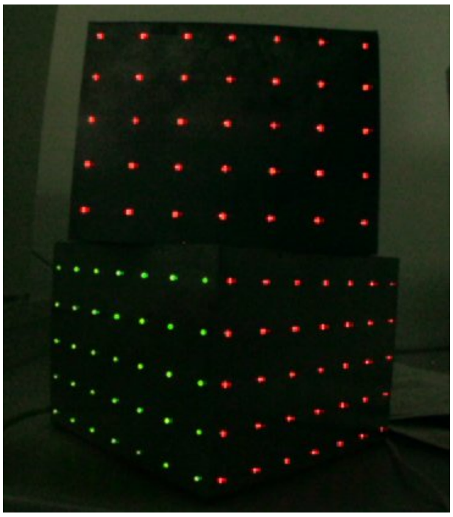
```
P = P./norm(P(3,1:3));

[K,R] = rq(P(:,1:3));
t = inv(K)*P(:,4);
C = -R'*t;
```

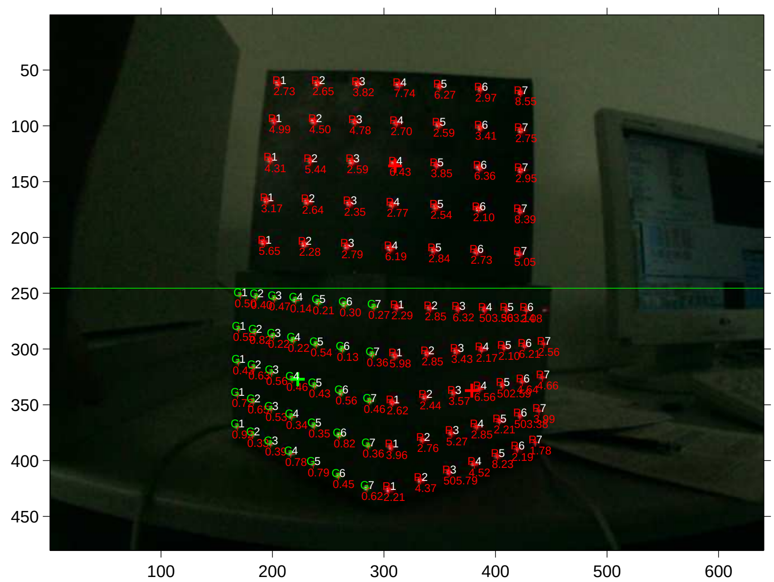
See the slide notes for more details.

An example of a calibration object

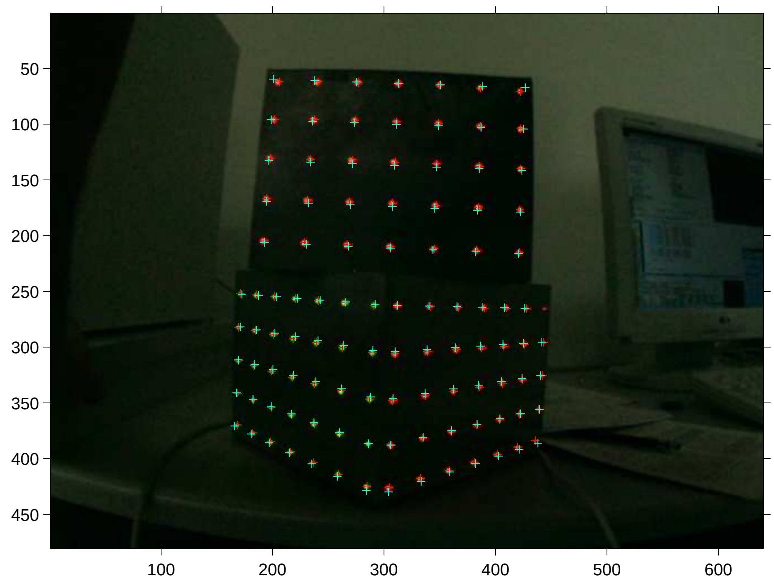




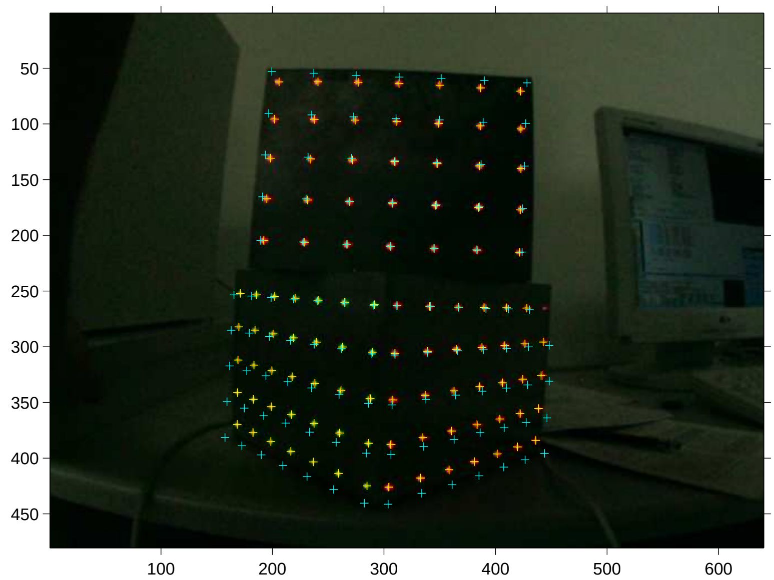
2D projections localized



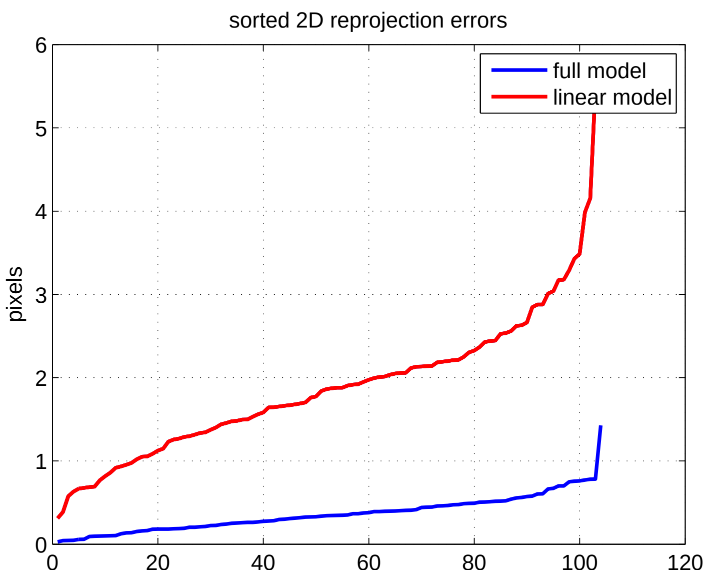
Reprojection for linear model



Reprojection for full model



Reprojection errors—comparison between full and linear model



References

The book [2] is the ultimate reference. It is a must read for anyone wanting use cameras for 3D computing.

Details about matrix decompositions used throughout the lecture can be found at [1]

[1] Gene H. Golub and Charles F. Van Loan. Matrix Computation. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, USA, 3rd edition, 1996.

[2] Richard Hartley and Andrew Zisserman. Multiple view geometry in computer vision. Cambridge University, Cambridge, 2nd edition, 2003.

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