Quantum logics representable as kernels of measures

Mirko Navara [1]

Abstract

By a kernel logic we mean a logic (=orthomodular poset) which is isomorphic to the kernel of some group-valued measure on a Boolean algebra. We suggest this structure as the base of a generalized probability theory for the description of systems including noncompatible events.

We exhibit the richness of the class of kernel logics by proving that it is closed under the formation of products, Boolean powers and horizontal sums. Every logic is a homomorphic image of a kernel logic.

We discuss the possibility of using measures with values in special groups.

We demonstrate the strength of our attitude by several quite nontrivial examples which were found desirable in previous works. Among others, we present a class of non-Boolean concrete logics with the Jauch-Piron property.

AMS Classification: 06C15, 03G12, 28B10.

Keywords: Concrete logic, orthomodular poset, group-valued measure, free product, horizontal sum, Boolean power, Jauch-Piron property, covering property.

1 Motivation

The classical Kolmogorov's model of probability assumes that every pair of events is simultaneously observable. This principle is violated in several applications including quantum mechanics, artificial intelligence, psychology, sociology etc. In these areas noncompatible events are encountered. These are events which can be observed separately, but not simultaneously, so they are not contained in a Boolean subalgebra (=classical subsystem) of the event structure describing the system in question. Various attempts have been made to generalize the probability theory to a more general structure admitting noncompatibility. Among them, classes of subsets (more generally, concrete logics) were studied for many years (see e. g. [15, 18]). Although some results were successfully generalized (see e. g. [4, 10, 19]), the theory proceeded slowly and with serious difficulties. Here we introduce a more special — but still reasonably general — structure, a kernel logic. As it is described in terms of Boolean algebras using measure-theoretic notions, we believe that there is a greater chance to generalize classical results for Boolean algebras to kernel logics.

Kernel logics seem to be interesting also from the algebraic point of view as a new construction technique for concrete logics. Its usefulness was proved by solutions of several quite nontrivial problems. Besides this, it seems desirable to describe kernels of measures on Boolean algebras.

[1] The author gratefully acknowledges the support of the EC grant PECO 3510 PL 922147 and of the grant no. 201/93/0953 of Grant Agency of Czech Republic

2 Basic definitions and examples

Let us recall the basic definitions (for more details, we refer to [6, 15]). By a *logic* we mean an orthomodular poset. A subset of a logic is called a *sublogic* if it is closed under orthocomplements and under orthogonal suprema. Let $\mathcal{K}, \mathcal{L}$ be logics. We call a mapping $h : \mathcal{K} \to \mathcal{L}$

- a *homomorphism* if it preserves the orthocomplements and orthogonal suprema,
- an *isomorphism* if it is one-to-one and both h, h^{-1} are homomorphisms,
- a *monomorphism* if $h : \mathcal{K} \to h(\mathcal{K})$ is an isomorphism.

In this paper, we shall mostly deal with the logics which are representable as collections of subsets of a set. Let X be a nonempty set. A collection $\mathcal{L} \subset 2^X$ is called a *class of subsets* if $X \in \mathcal{L}$ and $A, B \in \mathcal{L}$, $A \subset B$, implies $B \setminus A \in \mathcal{L}$. A class of subsets becomes a logic if we take the inclusion for the ordering and the complementation (in X) for the orthocomplementation (we use the notation $A^\perp = X \setminus A$). Notice that a class of subsets is closed under disjoint unions, but not under all unions. A logic $\mathcal{L}$ is called a *concrete logic* if it is isomorphic to a class $\mathcal{K}$ of subsets of a set. We call $\mathcal{K}$ a *representation* of $\mathcal{L}$. Of course, all Boolean algebras are concrete logics. A typical example of a non-Boolean concrete logic is the following.

Example 2.1 : Let $n, p \in N$ and let X be a set of cardinality $n \cdot p$. Then the collection $\mathcal{K}$ of all subsets of X whose cardinality is divisible by p is a class of subsets of X.

Let G be a commutative group. A *G-valued measure* on a logic $\mathcal{L}$ is a mapping $m : \mathcal{L} \to G$ such that $m(A \vee B) = m(A) + m(B)$ whenever $A \wedge B = 0$. A *two-valued measure* is a Z-valued measure with values $0, 1$.

Proposition 2.2 : *Let m be a G-valued measure on a Boolean algebra $\mathcal{B}$. Then the kernel of m,* $\operatorname{Ker} m = m^{-1}(0)$, *is a weak generalized orthomodular poset (see [7]). If, moreover, $m(1) = 0$, then* $\operatorname{Ker} m$ *is a concrete logic.*

Definition 2.3 : *A* kernel logic *is a logic which is isomorphic to* $\operatorname{Ker} m$ *for some group-valued measure m on a Boolean algebra.*

Remark 2.4 : *Throughout this paper we treat only the case $m(1) = 0$, leaving the investigation of weak generalized orthomodular posets (obtained for $m(1) \neq 0$) to another paper.*

Example 2.5 : The concrete logics from Ex. 2.1 are kernel logics. It suffices to take a measure $m : 2^X \to Z_p$ (Z_p is the p-element cyclic group) such that $m(\{x\}) = 1$ for all $x \in X$.

Concrete logics may be alternatively defined as orthomodular posets possessing order-determining sets of two-valued measures (i. e., for each $a, b,\ a \not\leq b$, there is a two-valued measure m such that $m(a) = 1 \neq m(b)$, see [20]). A concrete logic $\mathcal{L}$ may have various representations (see Ex. 4.1). It is reasonable to consider only such representations by a class $\mathcal{K}$ of subsets of a set X that, for each $x, y \in X$, there is an $A \in \mathcal{K}$ satisfying $x \in A$, $y \notin A$. The elements of X may be identified with two-valued measures on $\mathcal{L}$. (These measures correspond to concentrated measures on $\mathcal{K}$, i. e. to two-valued measures m such that $\exists x_m \in X\ \forall A \in \mathcal{K} : (m(A) = 1 \Leftrightarrow x_m \in A)$.) If X is the set of all two-valued measures on $\mathcal{L}$, we speak of a *maximal representation*.

An element a of a logic $\mathcal{L}$ is called an *atom* if $\{b \in \mathcal{L} : 0 < b < a\} = \emptyset$.

A finite subset $\mathcal{M}$ of a logic $\mathcal{L}$ is called *compatible* if it is contained in a Boolean algebra which is a sublogic of $\mathcal{L}$ (for more general and detailed exposition, see [15]). The *center* of $\mathcal{L}$ is the set $\{a \in \mathcal{L} : \{a, b\}$ is compatible for all $b \in \mathcal{L}\}$. The elements of the center are called *central*.

In the definition of a kernel logic, one may allow to express it in the form $\bigcap_{i\in I} \operatorname{Ker} m_i$, using a collection $\{m_i\}_{i\in I}$ of group-valued measures instead of a single measure. The following proposition shows that such a generalization does not bring anything new.

Lemma 2.6 : *Let $\mathcal{B}$ be a Boolean algebra and $\mathcal{L} \subset \mathcal{B}$ such that for each $A \in \mathcal{B} \setminus \mathcal{L}$ there is a commutative group G_A and a G_A-valued measure m_A on $\mathcal{B}$ satisfying $\mathcal{L} \subset \operatorname{Ker} m_A$, $A \notin \operatorname{Ker} m_A$. Then $\mathcal{L}$ is a kernel logic.*

Proof. We construct the product $G = \prod_{A\in\mathcal{B}\setminus\mathcal{L}} G_A$ and we define a measure $m : \mathcal{B} \to G$ by $m(C) = (m_A(C))_{A\in\mathcal{B}\setminus\mathcal{L}}$. Then $\operatorname{Ker} m = \bigcap_{A\in\mathcal{B}\setminus\mathcal{L}} \operatorname{Ker} m_A = \mathcal{L}$. □

3 Constructions with kernel logics

In order to find new examples of kernel logics, we discuss their relations to the basic constructions for orthomodular lattices — products, Boolean powers and horizontal sums. We prove that every logic is a homomorphic image of a kernel logic. For the description of products and horizontal sums of logics we refer to [6, 15], for Boolean powers in general to [3], in the context of logics to [2, 13].

Proposition 3.1 : *Every product of a family of kernel logics is a kernel logic.*

Proof. For $i \in I$, let $\mathcal{L}_i = \operatorname{Ker} m_i$, where m_i is a group-valued measure on a Boolean algebra $\mathcal{B}_i$. We define $\mathcal{B} = \prod_{i\in I} \mathcal{B}_i$ and $\mathcal{L} = \prod_{i\in I} \mathcal{L}_i \subset \mathcal{B}$, and we denote by $\pi_i : \mathcal{B} \to \mathcal{B}_i$ the cannonical projection. We shall apply Lemma 2.6 to prove that $\mathcal{L}$ is a kernel logic. If $A \in \mathcal{B} \setminus \mathcal{L}$ then there is an index $i \in I$ such that $\pi_i(A) \in \mathcal{B}_i \setminus \mathcal{L}_i = \mathcal{B}_i \setminus \operatorname{Ker} m_i$. The measure $m_i \circ \pi_i$ satisfies the assumption of Lemma 2.6. □

Proposition 3.2 : *Let $\mathcal{L}$ be a kernel logic, $\mathcal{A}$ a Boolean algebra. Then the bounded Boolean power $\mathcal{L}[\mathcal{A}]^*$ (as well as the Boolean power $\mathcal{L}[\mathcal{A}]$ provided that $\mathcal{A}$ is complete) is a kernel logic.*

Proof. We may assume that $\mathcal{A}$ is an algebra of subsets of a set Y. Let $\mathcal{L}$ be the kernel of a measure on a Boolean algebra $\mathcal{B}$. The bounded Boolean power $\mathcal{B}[\mathcal{A}]^*$ is a subset of $\prod_{y\in Y} \mathcal{B}_y$, where $\mathcal{B}_y \cong \mathcal{B}$ $(y \in Y)$. As $\mathcal{L}[\mathcal{A}]^* = \mathcal{B}[\mathcal{A}]^* \cap \prod_{y\in Y} \mathcal{L}_y$, where $\mathcal{L}_y \cong \mathcal{L}$, $\mathcal{L}_y \subset \mathcal{B}_y$ $(y \in Y)$, by the same technique as in the proof of Prop. 3.1 we may prove that $\mathcal{L}[\mathcal{A}]^*$ is a kernel of a measure on $\mathcal{B}[\mathcal{A}]^*$. □

Before treating horizontal sums, let us recall the construction of a free product (see [3] or [17], where it is called a "Boolean product"). Let $\{\mathcal{B}_i\}_{i\in I}$ be a collection of Boolean algebras. A *free product* of $\{\mathcal{B}_i\}_{i\in I}$ is a Boolean algebra $\mathcal{B}$ with monomorphisms $h_i : \mathcal{B}_i \to \mathcal{B}$ such that

1. If F is a finite subset of I and $A_i \in \mathcal{B}_i$, $A_i \neq 0$ $(i \in F)$, then $\bigwedge_{i\in F} h_i(A_i) \neq 0$.
2. $\bigcup_{i\in I} h_i(\mathcal{B}_i)$ generates $\mathcal{B}$.

The free product of a family of Boolean algebras always exists and it is unique up to an isomorphism; we denote it by $\mathbb{F}_{i\in I}\,\mathcal{B}_i$. It can be constructed from the set representations as follows. For each $i \in I$, let $\mathcal{B}_i$ be an algebra of subsets of a set X_i. Let X be the Cartesian product $\prod_{i\in I} X_i$ and let $p_i : X \to X_i$ be the cannonical projection. We define monomorphisms $h_i : \mathcal{B}_i \to 2^X$ by $h_i(A_i) = p_i^{-1}(A_i)$. (Thus, $h_i(A_i) = \prod_{j\in I} Y_j$, where $Y_i = A_i$ and $Y_j = X_j$ for $j \neq i$.) The algebra $\mathcal{B}$ of subsets of X, generated by $\bigcup_{i\in I} h_i(B_i)$, is the free product $\mathbb{F}_{i\in I}\,\mathcal{B}_i$. Notice that the free product is associative, i. e., if $J \subset I$, then $\mathbb{F}_{i\in I}\,\mathcal{B}_i$ is isomorphic to the free product of $\mathbb{F}_{i\in J}\,\mathcal{B}_i$ and $\mathbb{F}_{i\in I\setminus J}\,\mathcal{B}_i$.

For each $i \in I$, let m_i be a real-valued measure on $\mathcal{B}_i$ with $m_i(1_{\mathcal{B}_i}) = 1$. By a *product of measures* we mean the (unique) measure m (denoted also by $\prod_{i\in I} m_i$) on $\mathbb{F}_{i\in I}\,\mathcal{B}_i$ defined by the following rule: If F is a finite subset of I and $A_i \in \mathcal{B}_i$ $(i \in F)$, then $m(\bigwedge_{i\in F} h_i(A_i)) = \prod_{i\in F} m_i(A_i)$.

We first prove a special case.

Lemma 3.3 : *The horizontal sum of two kernel logics is a kernel logic.*

Proof. Let $\mathcal{L}$ be the horizontal sum of kernel logics $\mathcal{L}_i$, i=1,2. Let $\mathcal{L}_i = \operatorname{Ker} m_i$, where m_i is a G_i-valued measure on $\mathcal{B}_i$ and $\mathcal{B}_i$ is an algebra of subsets of a set X_i. We define $X = \prod_{i\in 1,2} X_i$, $\mathcal{B} = \mathbb{F}_{i=1,2}\,\mathcal{B}_i \subset 2^X$, and we denote by h_i the respective homomorphisms. We identify $\mathcal{L}$ with $\bigcup_{i=1,2} h_i(\mathcal{L}_i)$. According to Lemma 2.6, for each $A \in \mathcal{B} \setminus \mathcal{L}$ we have to find a group-valued measure m on $\mathcal{B}$ such that $\mathcal{L} \subset \operatorname{Ker} m$ and $m(A) \neq 0$. We shall distinguish two cases.

1. Let us suppose that $A \in \mathcal{B} \setminus \bigcup_{i=1,2} h_i(\mathcal{B}_i)$. As $A \notin h_1(\mathcal{B}_1)$, we may find $u_1 \in X_1$, $y_2, z_2 \in X_2$ such that $(u_1, y_2) \in A$, $(u_1, z_2) \notin A$. Analogously, as $A \notin h_2(\mathcal{B}_2)$, there are $u_2 \in X_2$, $y_1, z_1 \in X_1$ such that $(y_1, u_2) \in A$, $(z_1, u_2) \notin A$. For each point $(x_1, x_2) \in X$, we denote by $s_{(x_1,x_2)}$ the two-valued measure on $\mathcal{B}$ concentrated in (x_1, x_2). We define measures

$$\mu = s_{(u_1,u_2)} + s_{(y_1,y_2)} - s_{(u_1,y_2)} - s_{(y_1,u_2)},$$

$$\nu = s_{(u_1,u_2)} + s_{(z_1,z_2)} - s_{(u_1,z_2)} - s_{(z_1,u_2)}.$$

These measures vanish at $\bigcup_{i=1,2} h_i(\mathcal{B}_i)$, but at least one of them is nonzero on A. Indeed, the case $\mu(A) = \nu(A) = 0$ leads to a contradiction:

$$s_{(u_1,u_2)}(A) + s_{(y_1,y_2)}(A) - 2 = \mu(A) = \nu(A) = s_{(u_1,u_2)}(A) + s_{(z_1,z_2)}(A).$$

2. Suppose now that $A \in \bigcup_{i=1,2} h_i(\mathcal{B}_i) \setminus \mathcal{L}$. Without any loss of generality we may restrict our attention to the case $A \in h_1(\mathcal{B}_1 \setminus \mathcal{L}_1)$. Thus, $A = A_1 \times X_2$, where $A_1 \in \mathcal{B}_1 \setminus \mathcal{L}_1$. We fix a $y_2 \in X_2$ and we define a "line" $P = \{(x_1, x_2) \in X : x_2 = y_2\}$. The G_1-valued measure m on $\mathcal{B}$ defined by $m(C) = m_1(p_1(C \cap P))$, where $p_1 : X \to X_1$ is the cannonical projection, vanishes on $\mathcal{L}$, and $m(A) = m_1(A_1) \neq 0$. $\square$

Theorem 3.4 : *Every horizontal sum of kernel logics is a kernel logic.*

Proof. Let $\mathcal{L}$ be the horizontal sum of kernel logics $\mathcal{L}_i$, $i \in I$. For each $i \in I$, there is an algebra $\mathcal{B}_i$ of subsets of a set X_i and a group-valued measure m_i on $\mathcal{B}_i$ such that $\mathcal{L}_i = \operatorname{Ker} m_i$. Let $X = \prod_{i\in I} X_i$, $\mathcal{B} = \mathbb{F}_{i\in I}\, \mathcal{B}_i \subset 2^X$ and let h_i be the respective monomorphisms. We identify $\mathcal{L}$ with $\bigcup_{i\in I} h_i(\mathcal{L}_i) \subset \mathcal{B}$. We shall prove that $\mathcal{L}$ is a kernel logic.

Let $j, k \in I$, $j \neq k$, and denote by $p_{j,k} : X \to X_j \times X_k$ the cannonical projection. The class $\mathcal{L}_{j,k} = p_{j,k}(h_j(\mathcal{L}_j) \cup h_k(\mathcal{L}_k))$ of subsets of $X_j \times X_k$ is isomorphic to the horizontal sum of $\mathcal{L}_j$ and $\mathcal{L}_k$. For each $i \in I \setminus \{j, k\}$, we fix a $y_i \in X_i$. Consider a "plane"

$$\text{(P)} \qquad P = \{(x_i)_{i\in I} \in X : x_i = y_i \text{ for all } i \in I \setminus \{j,k\}\}.$$

If $A \in \mathcal{L}$, then $p_{j,k}(A \cap P) \in \mathcal{L}_{j,k}$ for all planes P of the form (P). The reverse implication is also true: If $A \notin \mathcal{L}$, there is a plane P of the form (P) (for suitably chosen j, k, y_i) such that $p_{j,k}(A \cap P) \notin \mathcal{L}_{j,k}$. According to Lemma 3.3, $\mathcal{L}_{j,k} = \operatorname{Ker} \mu$ for some group-valued measure μ. The measure m on $\mathcal{B}$ defined by $m(C) = \mu(p_{j,k}(C \cap P))$ vanishes at $\bigcup_{i\in I} h_i(\mathcal{L}_i)$ and $m(A) \neq 0$. According to Lemma 2.6, $\mathcal{L}$ is a kernel logic. □

Janowitz [5] introduced the class of *constructible lattices* — it is the smallest class of logics containing all Boolean algebras and closed under products and horizontal sums. Prop. 3.1 and Th. 3.4 have the following consequence.

Corollary 3.5 : *Every constructible logic is a kernel logic.*

The following theorem states that every orthomodular poset is a homomorphic image of a kernel logic. Moreover, we can require the homomorphism to "preserve compatibility".

Theorem 3.6 : *Let $\mathcal{L}$ be an orthomodular poset. There is a kernel logic $\mathcal{K}$ and a homomorphism $h : \mathcal{K} \xrightarrow{\text{onto}} \mathcal{L}$ such that:*

1. *the center of $\mathcal{L}$ is the image of the center of $\mathcal{K}$,*
2. *each finite compatible subset of $\mathcal{L}$ is an image of a compatible subset of $\mathcal{K}$.*

Proof. A concrete logic $\mathcal{K}$ with the latter properties is constructed in [1, 15, Th. 2.2.5]. It is obtained as the Boolean power of a horizontal sum of Boolean algebras, so it is a kernel logic (Prop. 3.2 and Th. 3.4). □

We must admit that, until now, we failed to find a concrete logic which is not a kernel logic. It seems that the answer to the question: "Is every concrete logic a kernel logic?" is either negative or rather nontrivial.

4 Kernels of measures with values in special groups

We may require to express a kernel logic as $\operatorname{Ker} m$, where m attains values in certain special group G. Despite some positive results, we shall show that this is not possible in general — for every group G there is a kernel logic which is not the kernel of a G-valued measure. Moreover, the choice of the set representation is also important, as we demonstrate by the following example.

Example 4.1 : 1. Consider the OML MO_3 (=the horizontal sum of 3 Boolean algebras 2^2, see [6]). It can be represented as the class $\mathcal{K}_1$ of subsets of a four-element set X_1 such that $\mathcal{K}_1 = \{A \subset X_1 : \operatorname{card} A$ is even$\}$. This is the special case of Ex. 2.1 for $n = p = 2$. According to Ex. 2.5, $\mathcal{K}_1 = \operatorname{Ker} m_1$ for a measure $m_1 : 2^{X_1} \to Z_2$. However, as a measure vanishing on $\mathcal{K}_1$ has to be constant on all singletons, $\mathcal{K}_1$ cannot be obtained as a kernel of a measure on 2^{X_1} with values in a group different from Z_2.

2. As MO_3 admits 8 two-valued measures, its maximal representation $\mathcal{K}_2$ has a domain X_2 with $\operatorname{card} X_2 = 8$. One may identify the elements of X_2 with the vertices of a cube so that $\mathcal{K}_2$ contains $\emptyset$, X_2, and each 4-element set of vertices corresponding to a surface of the cube. Then $\mathcal{K}_2 = \operatorname{Ker} m_2$ for a Z-valued measure m_2 on 2^{X_2} described by Fig. 1 (it was obtained by a simplified technique of Th. 3.4 and Prop. 4.3). However, $\mathcal{K}_2$ is not a kernel of a Z_2-valued measure on 2^{X_2}.

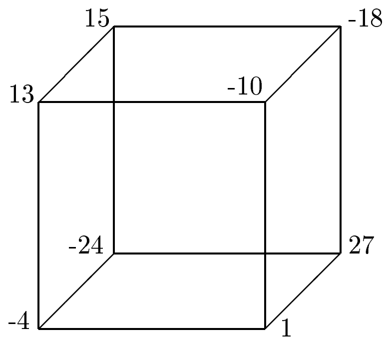

Fig. 1

3. Another set representation of MO_3 is the following: $X_3 = \{1, \ldots, 6\}$, $\mathcal{K}_3$ contains $\emptyset$, $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,5\}$, and the complements of these sets. The algebra of subsets of X_3 generated by $\mathcal{K}_3$ is 2^{X_3}. If m_3 is a measure on 2^{X_3} such that $\mathcal{K}_3 \subset \operatorname{Ker} m_3$, then

$$m_3(\{1,3,5\}) = m_3(\{1,2,3\}) + m_3(\{3,4,5\}) - m_3(\{2,3,4\}) = 0,$$

so $\{1,3,5\} \in \operatorname{Ker} m_3 \setminus \mathcal{K}_3$ and $\mathcal{K}_3$ is not the kernel of any group-valued measure on 2^{X_3}.

4. Checking all possible representations of MO_3, one may verify that there is no Z_3-valued measure whose kernel is isomorphic to MO_3.

The following theorem states that no group G is so "universal" to admit the description of all kernel logics as kernels of G-valued measures.

Theorem 4.2 : *For each commutative group G, there is a kernel logic $\mathcal{L}$ which is not isomorphic to the kernel of any G-valued measure.*

Proof. Let $\mathcal{L} = \prod_{i \in I} \mathcal{L}_i$, where $\mathcal{L}_i = MO_2 = \{0, a, b, a^\perp, b^\perp, 1\}$ $(i \in I)$ and $\operatorname{card} I > \operatorname{card} G$. According to Prop. 3.1 and Th. 3.4, $\mathcal{L}$ is a kernel logic.

Let X be a set of two-valued measures on $\mathcal{L}$, $\mathcal{K}$ a class of subsets of X representing $\mathcal{L}$ and $r : \mathcal{L} \to \mathcal{K}$ the cannonical isomorphism.

For each $e \in MO_2$, denote by e^i the element of $\mathcal{L}$ whose i-th coordinate is e, and all other coordinates are zero. For each $i \in I$, a^i, b^i are nonorthogonal atoms of $\mathcal{L}$ and there is only one two-valued measure, s_i, on $\mathcal{L}$ such that $s_i(a^i) = s_i(b^i) = 1$; this measure necessarily belongs to X. Analogously, X contains the measures t_i such that $t_i(a^i) = t_i((b^\perp)^i) = 1$ $(i \in I)$. Notice that each atom of $\mathcal{L}$ is represented by a two-element subset of X, e. g. $r(a^i) = \{s_i, t_i\}$.

Suppose that there is an algebra $\mathcal{B}$ of subsets of X and a measure $m : \mathcal{B} \to G$ such that $\mathcal{K} \subset \operatorname{Ker} m$. Then $\mathcal{B}$ contains the Boolean subalgebra generated by $\mathcal{K}$ and, in particular, all finite subsets of $\{s_i, t_i : i \in I\}$. Due to cardinality reasons, we have $m(\{s_i\}) = m(\{s_j\})$ for some $i, j \in I$, $i \neq j$. This implies $m(\{s_i, t_j\}) = m(\{s_j, t_j\}) = m(r(a^j)) = 0$, so $\{s_i, t_j\} \in \operatorname{Ker} m$. If $\{s_i, t_j\} \in \mathcal{K}$, then, as 1^i is a central element of $\mathcal{L}$, $\{s_i\} = \{s_i, t_j\} \cap r(1^i) \in \mathcal{K}$, and $\{s_i\}$ becomes an atom in $\mathcal{K}$ which is central — a contradiction. So $\mathcal{K} \subsetneq \operatorname{Ker} m$. □

There are still some important cases in which measures with values in certain groups are sufficient for the description of a class of kernel logics. For instance, for finite logics we can strengthen Cor. 3.5:

Proposition 4.3 : *Every finite constructible logic is the kernel of some integer-valued measure.*

Proof. The technique of the proofs of Prop. 3.1 and Th. 3.4 results in a set of Z-valued measures $m_1, \ldots, m_n$ such that $\mathcal{L} = \bigcap_{i \leq n} \operatorname{Ker} m_i$. There is an $M \in N$ such that the values of m_i, $i \leq n$, do not exceed the interval $(-M, M)$. Then $m = \sum_{i \leq n} M^i m_i$ is a Z-valued measure with $\mathcal{L} = \operatorname{Ker} m$. □

5 Applications to "almost Boolean" logics

In the study of classes of subsets, we often have to investigate a class $\mathcal{K}$ of subsets of a set X such that $\mathcal{K}$ contains a given collection $\mathcal{M} \subset 2^X$. It is usually difficult to determine the class of subsets generated by $\mathcal{M}$ (or, at least, to find a "small" class of subsets containing $\mathcal{M}$). Sometimes it took many years until the structure of a specific class of subsets was clarified and nontrivial combinatorial reasoning has been utilized (see e. g. [10, 12, 19]). A collection of such problems appeared in the study of concrete logics which have some properties similar to those of Boolean algebras (see [9, 11]). We show here that the answers can be efficiently obtained and described by means of kernel logics. In this approach, one finds an appropriate measure (or a collection of measures, see Prop. 2.6), the kernel of which contains $\mathcal{M}$. It becomes quite easy to check which sets belong to the corresponding class of subsets. We present here several applications of this attitude.

A logic $\mathcal{L}$ has the *Jauch-Piron property* ([16]) if, for each **non-negative** finite real-valued measure s, each $A, B \in \operatorname{Ker} s$ have an upper bound $C \in \operatorname{Ker} s$ (i. e., if $\operatorname{Ker} s$ with $\subset$ is a directed set). We denote by $\mathcal{CL}_{JP}$ the class of concrete logics with the Jauch-Piron property and by $\mathcal{BA}$ the class of Boolean algebras. Obviously, $\mathcal{BA} \subset \mathcal{CL}_{JP}$. The question has arisen whether the latter inclusion is proper. This problem was formulated e.g. in [11, 14] and remained open for several years. An affirmative answer was given in [8]. Here we provide a family of examples of kernel logics from $\mathcal{CL}_{JP} \setminus \mathcal{BA}$. We shall make use of the following lemma which is mentioned, without proof, in [8].

Lemma 5.1 : *Let $\mathcal{K}$ be a class of subsets satisfying the following property:*

> (M) *For each $A, B \in \mathcal{K}$ there are uncountable families $(C_t)_{t \in T}$, $(D_t)_{t \in T}$ of elements of $\mathcal{L}$ such that $(C_t)_{t \in T}$ is disjoint and $C_t \cup D_t = A \cap B$ $(t \in T)$.*

Then $\mathcal{K}$ has the Jauch-Piron property.

Proof. Let m be a measure on $\mathcal{K}$ such that $m(A^\perp) = m(B^\perp) = 0$. As T is uncountable, $m(C_u) = 0$ for some $u \in T$. As $D_u^\perp \setminus B^\perp \subset A^\perp \cup C_u \in \operatorname{Ker} m$, we obtain $m(D_u^\perp) = 0$ for $D_u^\perp \supset A^\perp \cup B^\perp$. □

Example 5.2 : There are kernel logics which are not Boolean algebras and satisfy the Jauch-Piron property.

Let W be the union of two disjoint uncountable sets U, V. We denote by $\mathcal{C}$ the Boolean algebra of all finite and cofinite subsets of W. Measure $\mu : \mathcal{C} \to Z$ is uniquely determined by the following rules:

$\mu(\{u\}) = 1$ for all $u \in U$,

$\mu(\{v\}) = -1$ for all $v \in V$,

$\mu(W) = 1$.

Take an infinite set I and one another element, say $1 \notin I$. We construct the free product $\mathcal{B} = \mathbb{F}_{i \in I_1} \mathcal{B}_i$, where $I_1 = \{1\} \cup I$ and $\mathcal{B}_i = \mathcal{C}$ $(i \in I_1)$. We denote by $h_i : \mathcal{B}_i \to \mathcal{B}$ the corresponding monomorphisms. We define measures $m_i : \mathcal{B}_i \to Z$ so that $m_i = \mu$ for all $i \in I_1$. Let $\rho_1 : \mathcal{B}_1 \to Z$ be the two-valued measure attaining 1 exactly on all cofinite sets. We define a measure $m : \mathcal{B} \to Z$ by the formula

$$m = \prod_{i \in I_1} m_i - \rho_1 \cdot \prod_{i \in I} m_i.$$

We claim that $\mathcal{L} = \operatorname{Ker} m$ has the required properties. (The subtraction of ρ_1 ensured that $m(1_{\mathcal{B}}) = 0$. In fact, m can be constructed as a product $\nu_1 \cdot \prod_{i \in I} m_i$, where $\nu_1 = m_1 - \rho_1$. However, we could not apply immediately the standard construction because $\nu_1(1_{\mathcal{B}_1}) = 0 \neq 1$.)

To see that $\mathcal{L}$ is not Boolean, take $u \in U$, $v', v'' \in V$, $v' \neq v''$. Then $h_1(\{u\}) \cup h_1(\{v'\})$, $h_1(\{u\}) \cup h_1(\{v''\}) \in \mathcal{L}$, while their intersection $h_1(\{u\}) \notin \mathcal{L}$.

It remains to prove that the condition (M) of Lemma 5.1 is satisfied. Let $A, B \in \mathcal{L}$ and let T be a set of the first uncountable cardinality. If $A \cap B \in \mathcal{L}$, we may choose $C_t = 0$, $D_t = A \cap B$. If $A \cap B \notin \mathcal{L}$, $A \cap B$ contains a subset of the form $E = \bigcap_{i \in F} h_i(\{e_i\})$, where F is a finite subset of I_1 and $e_i \in W$. Notice that $m(E) = \pm 1$. Put $n = m(A \cap B) \in Z \setminus \{0\}$. We fix a $j \in I \setminus F$ and we choose mutually disjoint sets $U_t \subset U$, $V_t \subset V$ with $\operatorname{card} U_t = \operatorname{card} V_t = |n|$ $(t \in T)$. It suffices to take $C_t = E \cap h_j(U_t \cup V_t)$, $D_t = (A \cap B) \setminus (E \cap h_j(Y_t))$, where $Y_t = U_t$ if $m(A \cap B) \cdot m(E) > 0$ and $Y_t = V_t$ otherwise. Lemma 5.1 finishes the proof.

Remark 5.3 : *We have constructed a collection of new examples. The original example of Müller [8] is a proper sublogic of each of these.*

Let $n \in N \cup \{\infty\}$ and let $\mathcal{K}$ be a class of subsets. Following [9], we say that $\mathcal{K}$ belongs to the class $\mathcal{CL}_n$ if, for each $A, B \in \mathcal{K}$, there exist $C_i \in \mathcal{K}$, $i < n$, such that $A \cap B = \bigcup_{i<n} C_i$. Obviously, $\mathcal{CL}_2 = \mathcal{BA}$. It was proved in [11] that $\mathcal{CL}_{JP} \subsetneqq \mathcal{CL}_3$. Again the question remained open for some time whether the inclusions $\mathcal{CL}_n \subset \mathcal{CL}_{n+1}$, $n \in N$, $n \geq 3$, are proper. An affirmative answer was given in [9]. Here we construct a family of examples which are kernel logics.

Example 5.4 : Let $n \in N$, $n \geq 2$. Then there is a kernel logic $\mathcal{L} \in \mathcal{CL}_{n+1} \setminus \mathcal{CL}_n$.

Let $\mathcal{C} = 2^{\{0,\ldots,n\}}$ and let $\mu : \mathcal{C} \to Z$ be a measure such that

$\mu(\{0\}) = -1,$

$\mu(\{k\}) = 1,\ k = 1, \ldots, n.$

Notice that $\mu(\{0, \ldots, n\}) = n-1$ and $1_{\mathcal{C}} = \{0, \ldots, n\}$ can be covered by n elements of $\operatorname{Ker}\mu$ (namely, $\{0, k\}$, $k = 1, \ldots, n$), but not by $n-1$ elements.

Take an infinite set I and one another element, say $1 \notin I$. We construct the free product $\mathcal{B} = \mathbb{F}_{i \in I_1} \mathcal{B}_i$, where $I_1 = \{1\} \cup I$, $\mathcal{B}_i = \mathcal{C}$ $(i \in I)$ and $\mathcal{B}_1 = 2^{\{0,1,2,3\}}$. We denote by $h_i : \mathcal{B}_i \to \mathcal{B}$ $(i \in I_1)$ the corresponding monomorphisms. For all $i \in I$, we define measures $m_i : \mathcal{B}_i \to R$ so that $m_i = \mu/(n-1)$ (hence $m_i(1_{\mathcal{B}_i}) = 1$). Let m_1 be the Z-valued measure on $\mathcal{B}_1$ such that $m_1(\{k\}) = (-1)^k$, $k = 0, \ldots, 3$. We would like to use the product of measures m_i, $i \in I_1$, but again, as $m_1(1_{\mathcal{B}_1}) = 0$, we must express m_1 in the form $m_1^+ - m_1^-$, where $m_1^+(1_{\mathcal{B}_1}) = m_1^-(1_{\mathcal{B}_1}) = 1$. We may choose e. g. $m_1^+(\{k\}) = (1+2\cdot(-1)^k)/4$, $m_1^-(\{k\}) = (1-2\cdot(-1)^k)/4$. We define a measure $m : \mathcal{B} \to R$ by the formula

$$m = m_1^+ \cdot \prod_{i \in I} m_i - m_1^- \cdot \prod_{i \in I} m_i.$$

We claim that $\mathcal{L} = \operatorname{Ker} m$ has the required properties.

Let C be the intersection of two elements of $\mathcal{L}$. As $C \in \mathcal{B}$, it can be expressed as a finite union of disjoint nonempty sets of the form $E = \bigcap_{i \in F} h_i(A_i)$, where F is a finite subset of I_1 and $A_i \in \mathcal{B}_i$. We fix $j \in I \setminus F$. Then

$$E = E \cap h_j(\{0, \ldots, n\}) = \bigcup_{k=1,\ldots,n} (E \cap h_j(\{0, k\})),$$

where $E \cap h_j(\{0, k\}) \in \mathcal{L}$. The same procedure being applied to all (disjoint) parts of C, we obtain a covering of C by n elements of $\mathcal{L}$.

It remains to prove that $\mathcal{L} \notin \mathcal{CL}_n$. Take $A = h_1(\{0,1\}) \in \mathcal{L}$, $B = h_1(\{0,3\}) \in \mathcal{L}$. Then $C = A \cap B = h_1(\{0\})$ cannot be covered by less than n elements of $\mathcal{L}$.

Remark 5.5 : *For each $n \in N$, we have got a collection of examples different from that of [9]. Unlike to the approach applied there, we have a unique technique for all $n \in N$.*

Example 5.6 : There is a kernel logic $\mathcal{L} \in \mathcal{CL}_\infty \setminus \bigcup_{n \in N} \mathcal{CL}_n$.

One may apply the same technique as in Ex. 5.4. The only difference is that we take $I = N \setminus \{1\}$ and $\mathcal{B}_i = 2^{\{0,\ldots,i\}}$ for $i \in I$. The measures m_i, $i \in I$, are again normalized so that $m_i(1_{\mathcal{B}_i}) = 1$.

As a corollary of the above facts, we obtain the following proper inclusions [9]: $\mathcal{BA} = \mathcal{CL}_2 \subsetneqq \mathcal{CL}_{JP} \subsetneqq \mathcal{CL}_\infty$, $\mathcal{CL}_3 \subsetneqq \mathcal{CL}_4 \subsetneqq \ldots \subsetneqq \mathcal{CL}_\infty$. It is not known whether $\mathcal{CL}_{JP} \subset \mathcal{CL}_n$ for some finite $n \geq 3$. According to Ex. 5.4, $\mathcal{CL}_3 \not\subset \mathcal{CL}_{JP}$.

References

[1] Binder, J.: A Loomis-Sikorski theorem for logics. *Math. Slovaca* **38** (1988), 367–371.

[2] Drossos, C.A.: Boolean methods and Pták's sum. To appear.

[3] Grätzer, G.: *Universal Algebra*. Springer-Verlag, New York/ Heidelberg/Berlin, 1979.

[4] Gudder, S.P., Zerbe, J.: Generalized monotone convergence and Radon-Nikodym theorems. *J. Math. Physics* **22** (1981), 2553–2561.

[5] Janowitz, M.F.: Constructible lattices. *J. Austr. Math. Soc.* **14** (1972), 311–316.

[6] Kalmbach, G.: *Orthomodular Lattices*. Academic Press, London, 1983.

[7] Mayet-Ippolito, A.: Generalized orthomodular posets. *Demonstratio Math.* **24** (1991), 263–274.

[8] Müller, V.: Jauch-Piron states on concrete quantum logics. *Int. J. Theor. Phys.* **32** (1993), 433–442.

[9] Müller, V., Pták, P., Tkadlec, J.: Concrete quantum logics with covering properties. *Int. J. Theor. Phys.* **31** (1992), 843–854.

[10] Navara, M.: When is the integral on quantum probability spaces additive?. *Real Analysis Exchange* **14** (1989), 228–234.

[11] Navara, M., Pták, P.: Almost Boolean orthomodular posets. *J. Pure Appl. Algebra* **60** (1989), 105–111.

[12] Olejček, V.: Generation of a q-σ-algebra in the plane. *Proc. Conf. Topology and Measure* **V**, Greifswald, 121–125, 1988.

[13] Pták, P.: Summing of Boolean algebras and logics. *Demonstratio Math.* **19** (1986), 349–358.

[14] Pták, P.: FAT $\leftrightarrow$ CAT (in the state space of quantum logics). *Proc. 1st Winter School on Measure Theory*, Liptovský Ján, 113–118, 1988.

[15] Pták, P., Pulmannová, S.: *Orthomodular Structures as Quantum Logics*. Kluwer Academic Publishers, Dordrecht/Boston/London, 1991.

[16] Rüttimann, G.T.: Jauch-Piron states. *J. Math. Phys.* **18** (1977), 189–193.

[17] Sikorski, R.: *Boolean Algebras*. Springer-Verlag, Berlin, 1969.

[18] Suppes, P.: The probabilistic argument for a nonclassical logic of quantum mechanics. *Philos. Sci.* **33** (1966), 14–21.

[19] Zerbe, J.E., Gudder, S.P.: Additivity of integrals on generalized measure spaces. *J. Comb. Theory* (A) **30** (1985), 42–51.

[20] Zierler, N., Schlessinger, M.: Boolean embeddings of orthomodular sets and quantum logic. *Duke Math. J.* **32** (1965), 251–262.

Department of Mathematics, Faculty of Electrical Engineering,
Czech Technical University, 166 27 Praha 6, Czech Republic