Constraint on Five Points in Two Images

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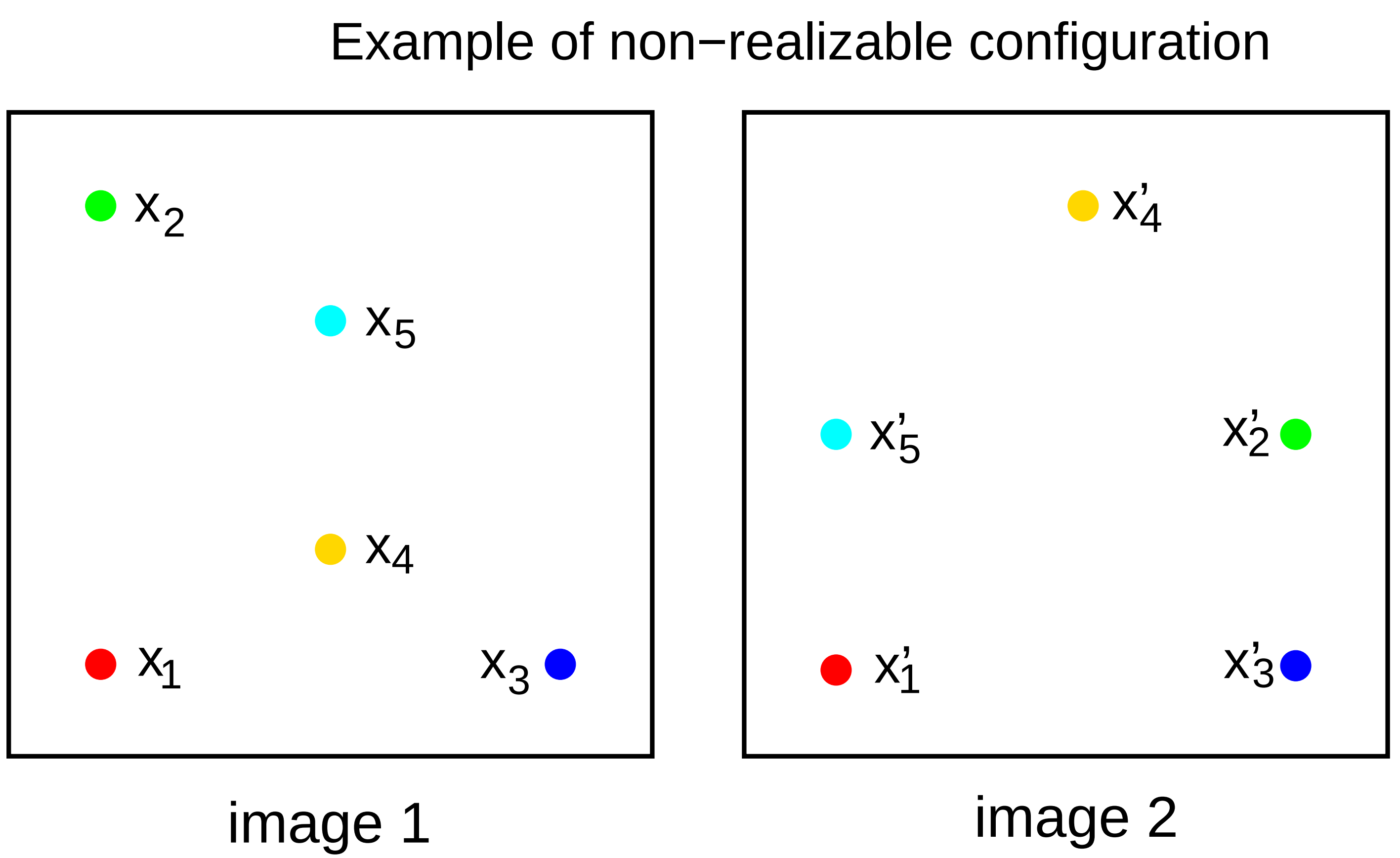
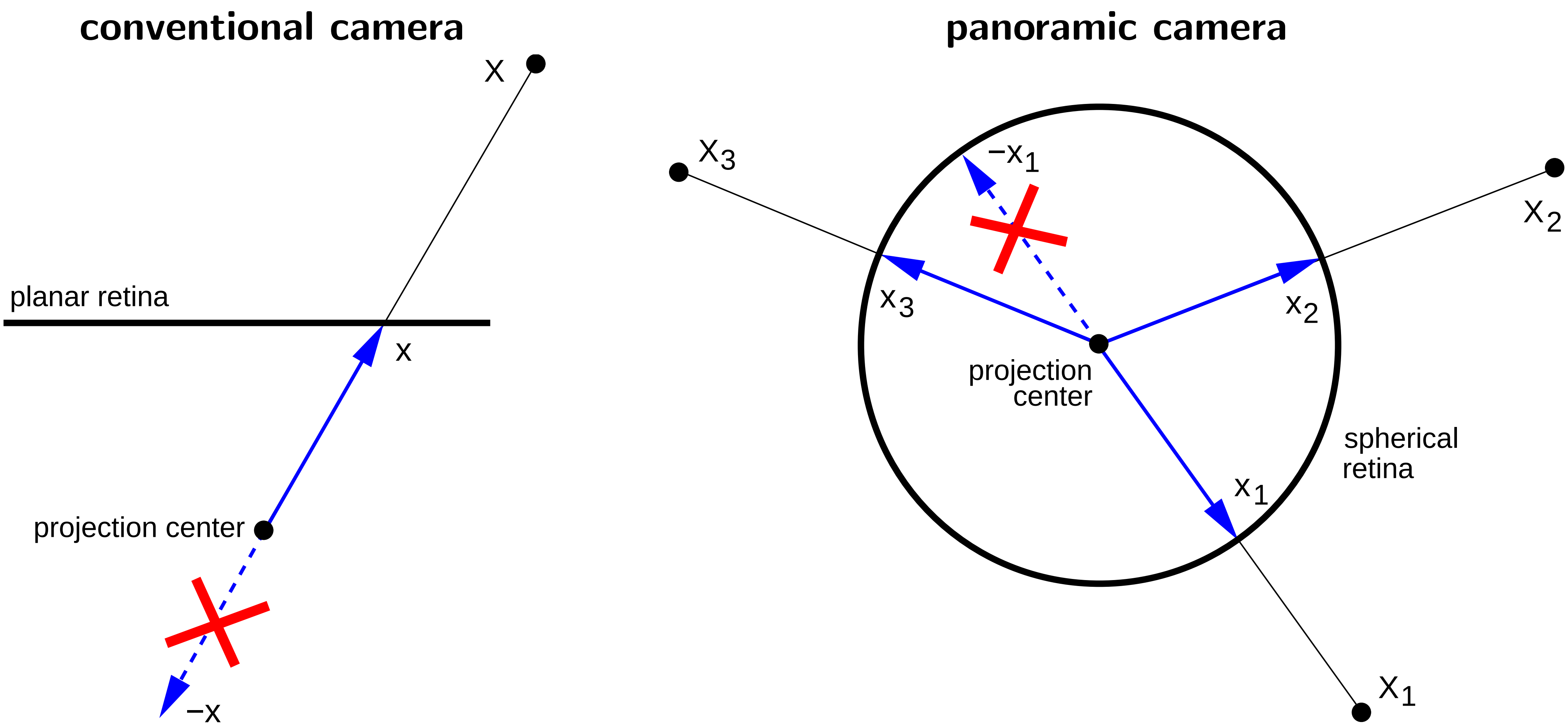
There exist configurations of five pairs of corresponding points in two images that are not projections of any five scene points by any pair of pinhole cameras.



What Is Behind: Orientation of Camera Rays

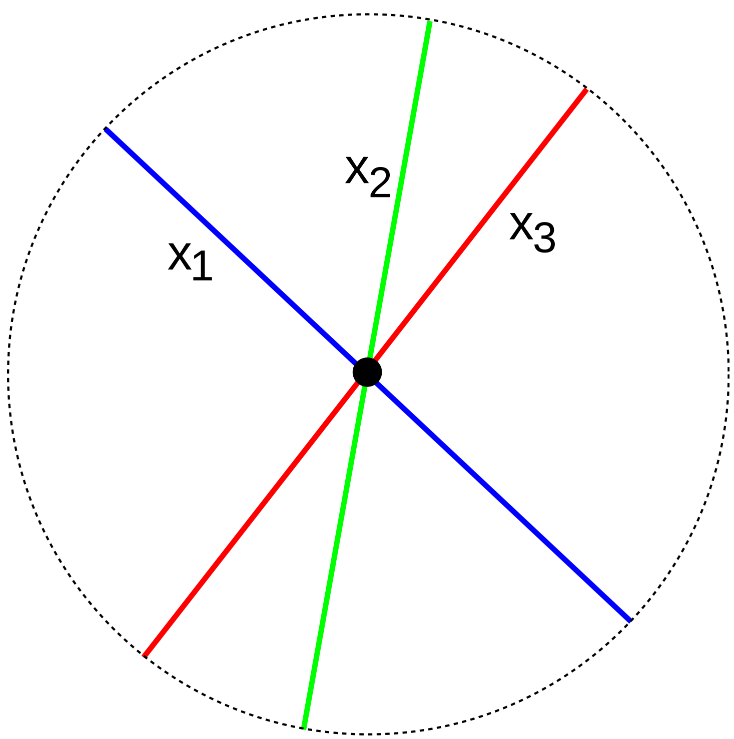


correct choice: $\mathbf{x} = (x, y, 1)^\top$

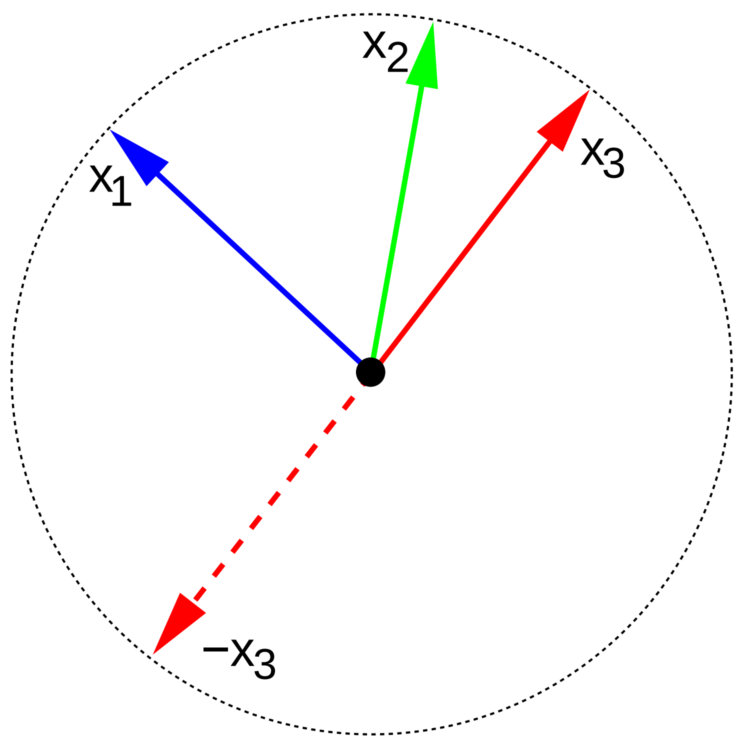
- Points on wrong sides of camera rays are forbidden.
- In $\lambda\mathbf{x} = \mathbf{P}\mathbf{X}$, it must be $\lambda > 0$.

Projective vs. Spherical Space

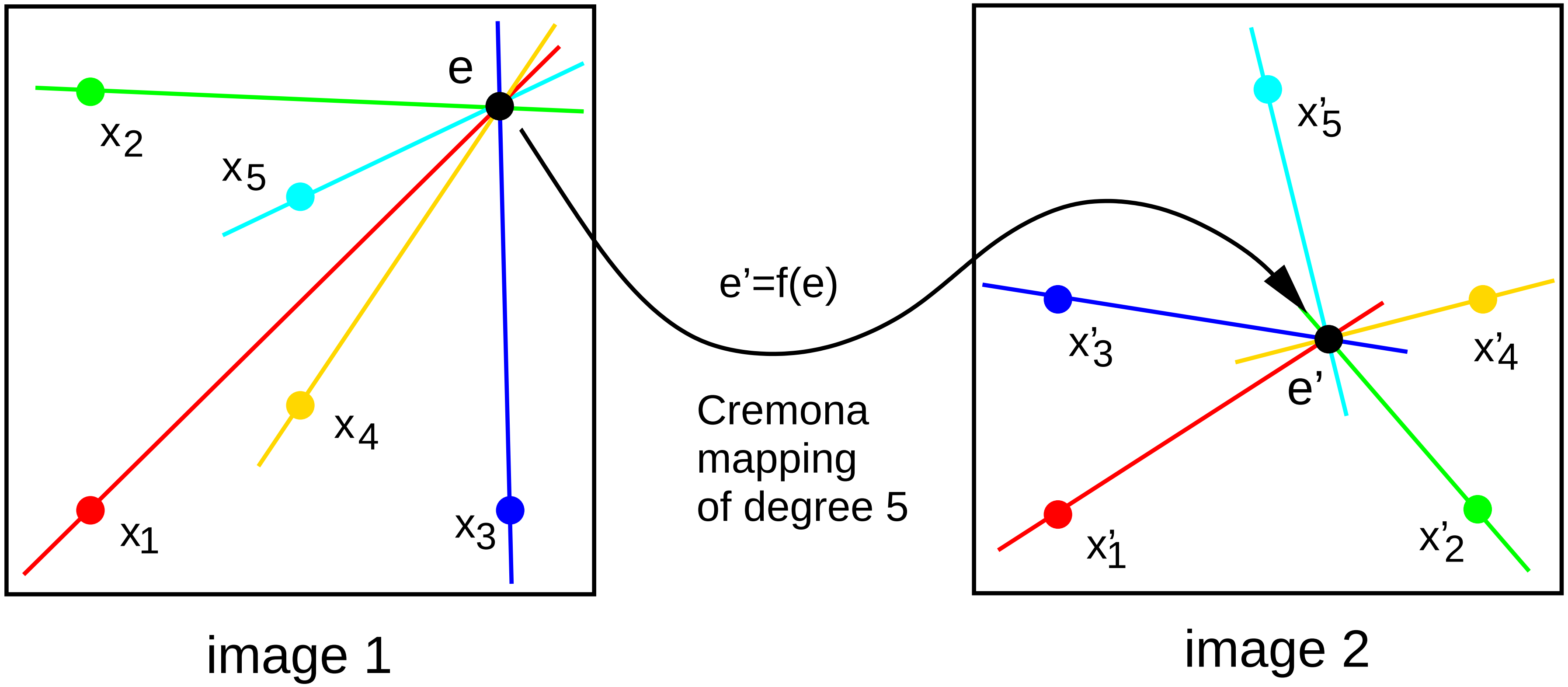
projective space $\mathbb{P}_1$
(straight lines through a point)



spherical space $\mathbb{S}_1$
(halflines through a point)



Spherical 2-space $\mathbb{S}^2$ is a better model of camera retina than projective plane $\mathbb{P}^2$ [Stolfi-91,Laveau-Faugeras-94,Werner-Pajdla-2001].



Line pencil through e and line pencil through e' are related by a homography:

$$e(x_1,\ldots,x_5)\ \overline{\wedge}\ e'(x'_1,\ldots,x'_5)$$

$\Rightarrow$ $\boxed{e' = f(e) \text{ where } f \text{ is a Cremona mapping of degree 5 [Sturm-1908]}}$

Orientation Constraint on Epipoles

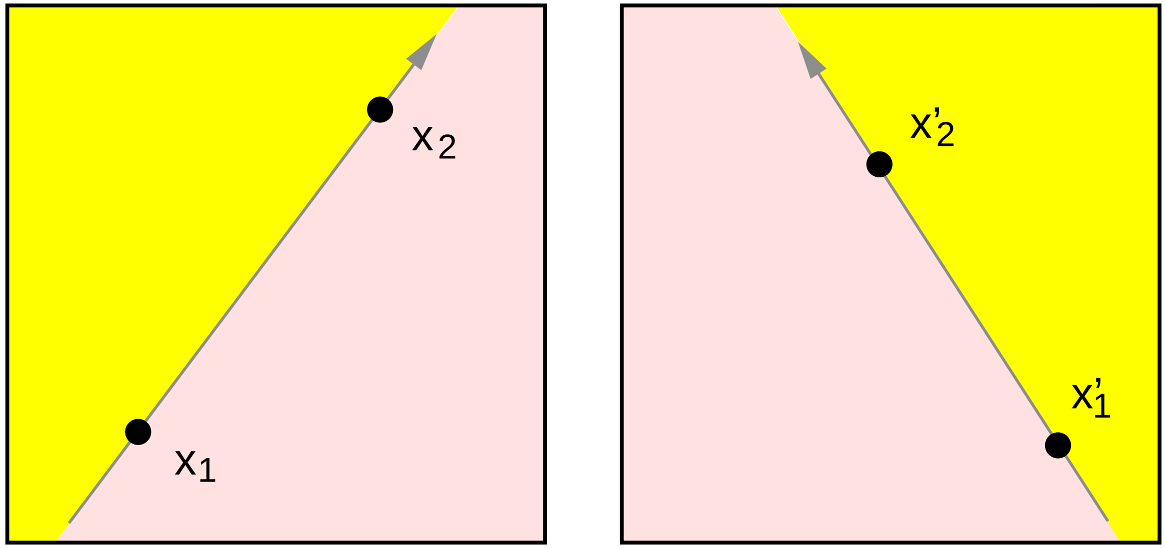
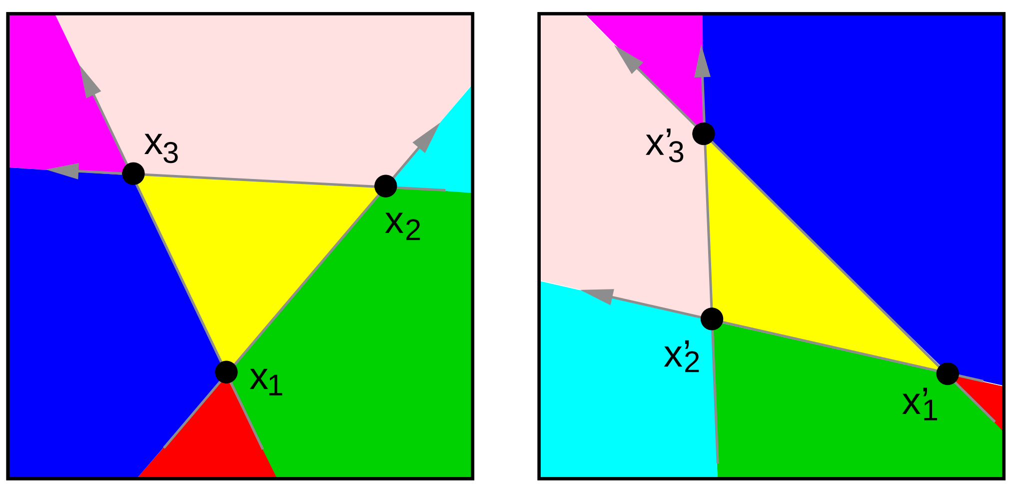


image 1 image 2

e lies on RHS of oriented line $x_1 x_2$ $\Leftrightarrow$ e' lies on LHS of oriented line $x'_1 x'_2$
and *vice versa* [Werner-Pajdla-2001]

For more points:



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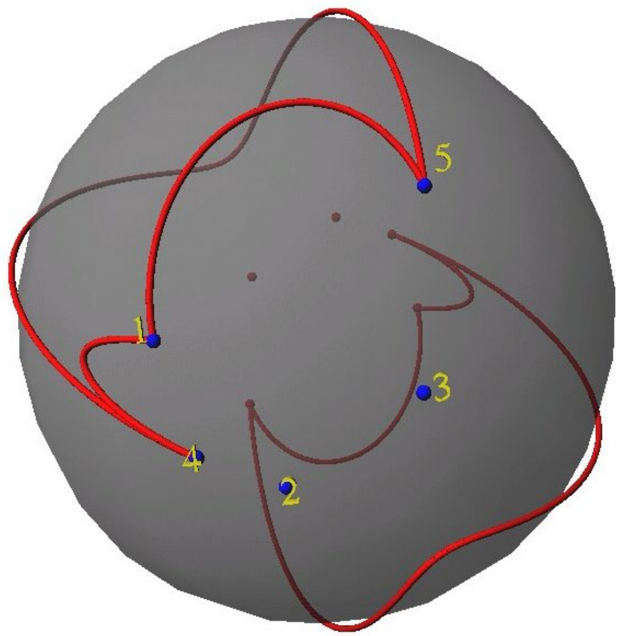
Combining the Two Constraints

Combining projective and orientation constraint $\Rightarrow$ the result of this paper



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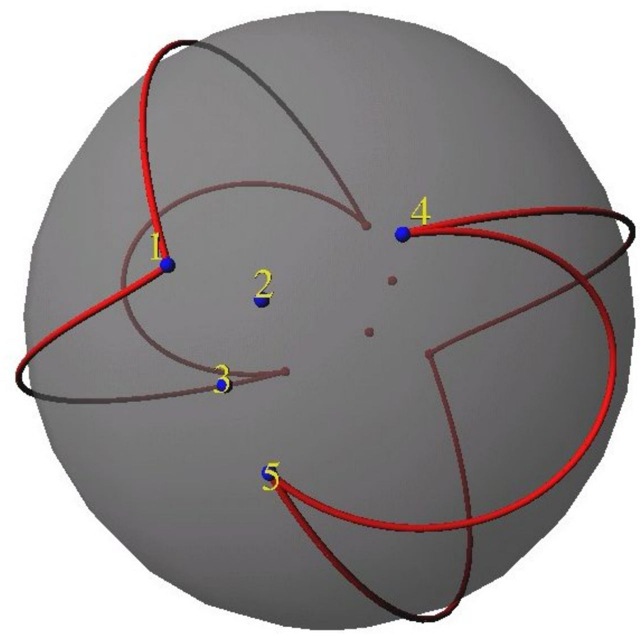

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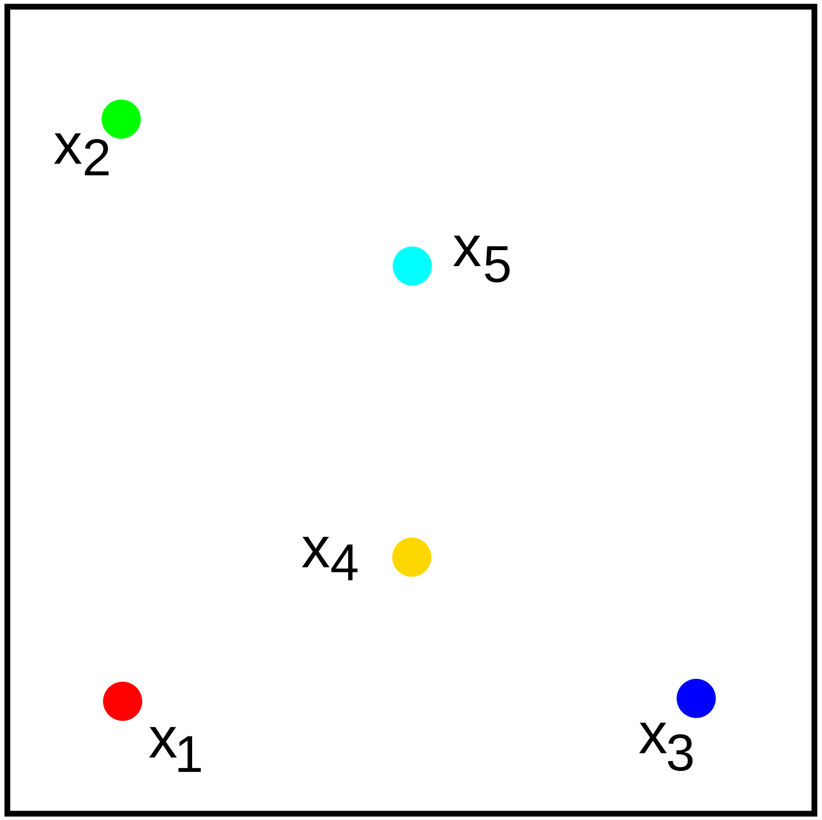
- Allowed epipoles lie in domains with piecewise-conic boundaries.
- For non-realizable configurations, these domains are empty.

Where Can Be the Last Point?



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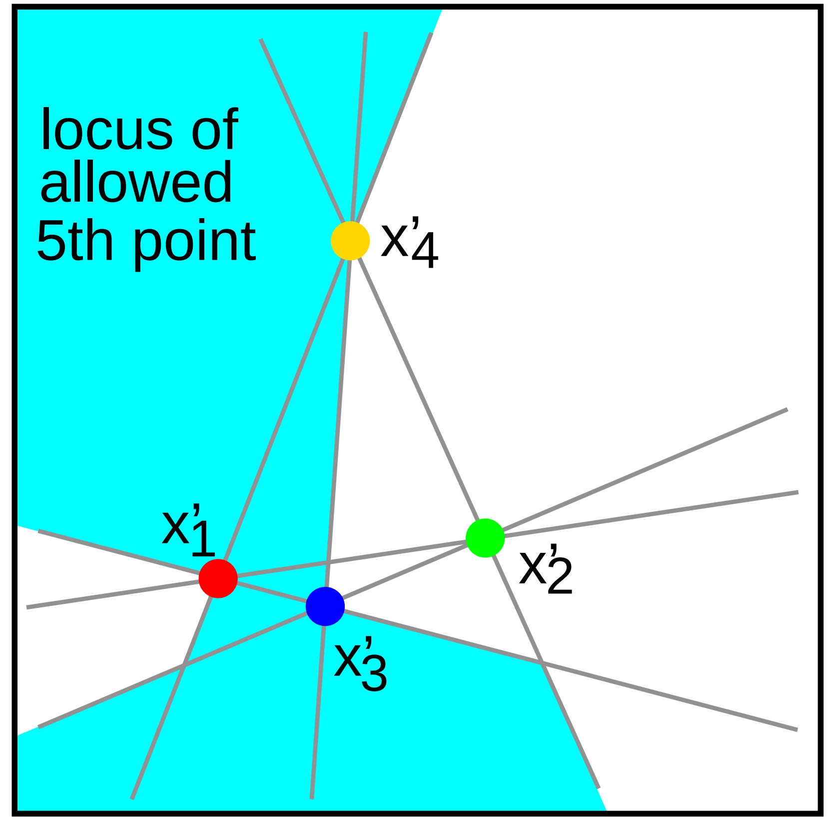


image 2

Fix all 10 points of the 5 pairs except the last one. Then,

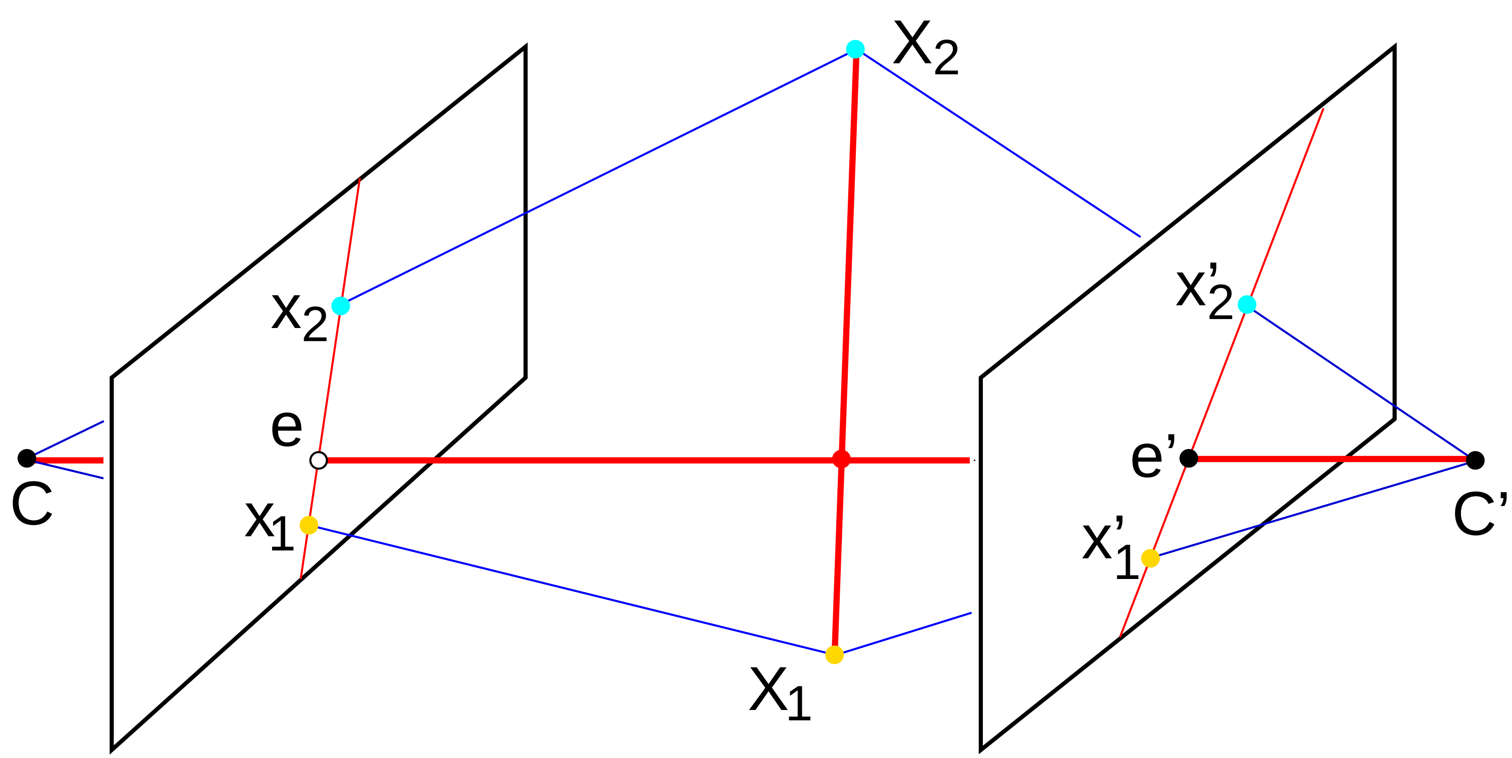
> the allowed positions of the point form a non-convex polygon.

Conclusion

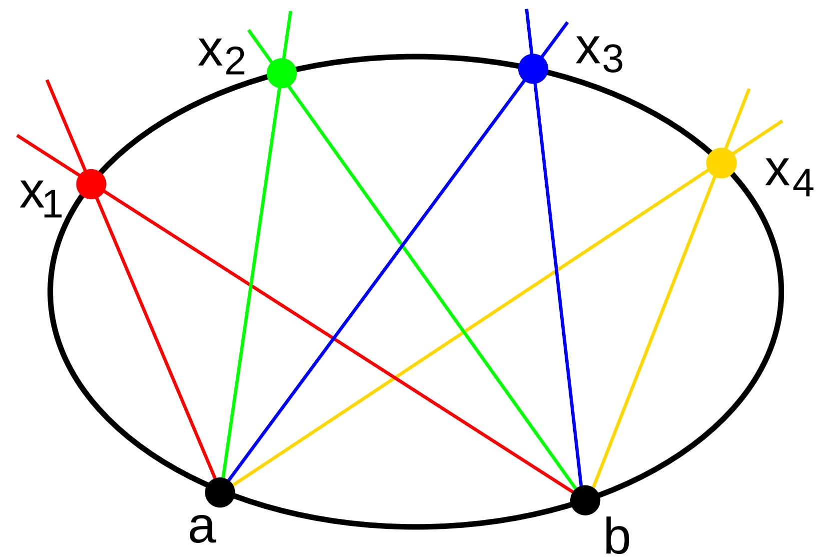
- Some configurations of 5 point correspondences in 2 images are not projections of any scene points by any cameras.
- No assumptions on cameras at all. All real cameras behave like this.
- Why not noticed before?
  - Orientation consistency has been ignored.
  - Non-trivial derivation.
- Consequences for other topics from multiple view geometry $\Rightarrow$ future work.

Projective Constraint for $n = 2$

- Lines joining the points are preserved: $\boxed{(e \in x_1 x_2) \ \Leftrightarrow \ (e' \in x'_1 x'_2).}$
- Otherwise, (e, e') can be arbitrary.



[Steiner]: a, b, x_1, x_2, x_3, x_4 are conconic $\Leftrightarrow$ $a(x_1, x_2, x_3, x_4) \barwedge b(x_1, x_2, x_3, x_4)$



Hence, e is free and e' is restricted to conic $h(Q^e)$:

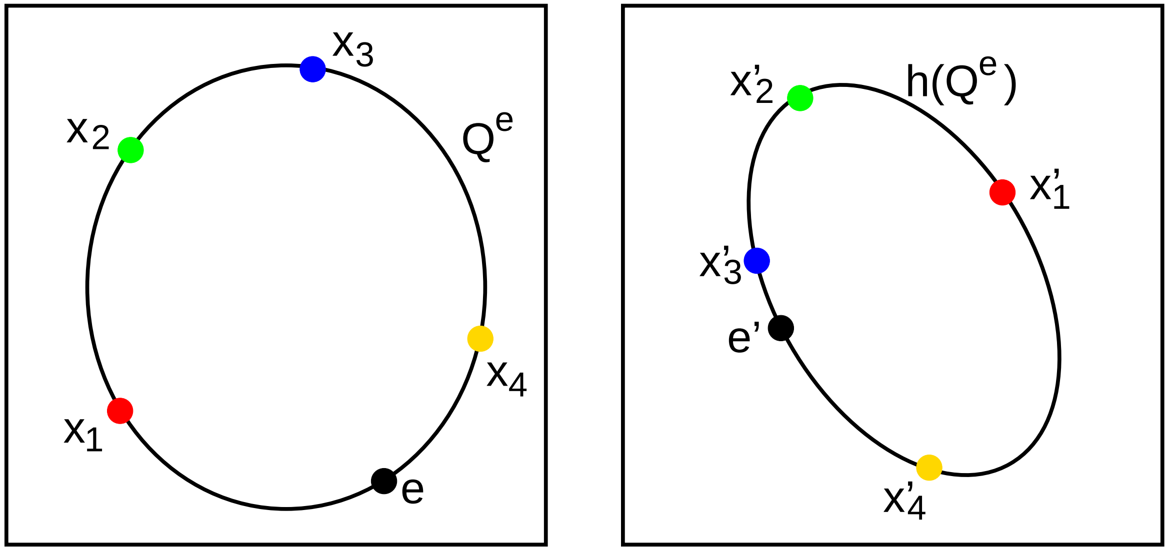


image 1 image 2

Projective Constraint for $n = 5$

The four-point constraint must hold simultaneously for each 4-tuple of correspondences.

E.g., for 4-tuples 2345 and 1345:

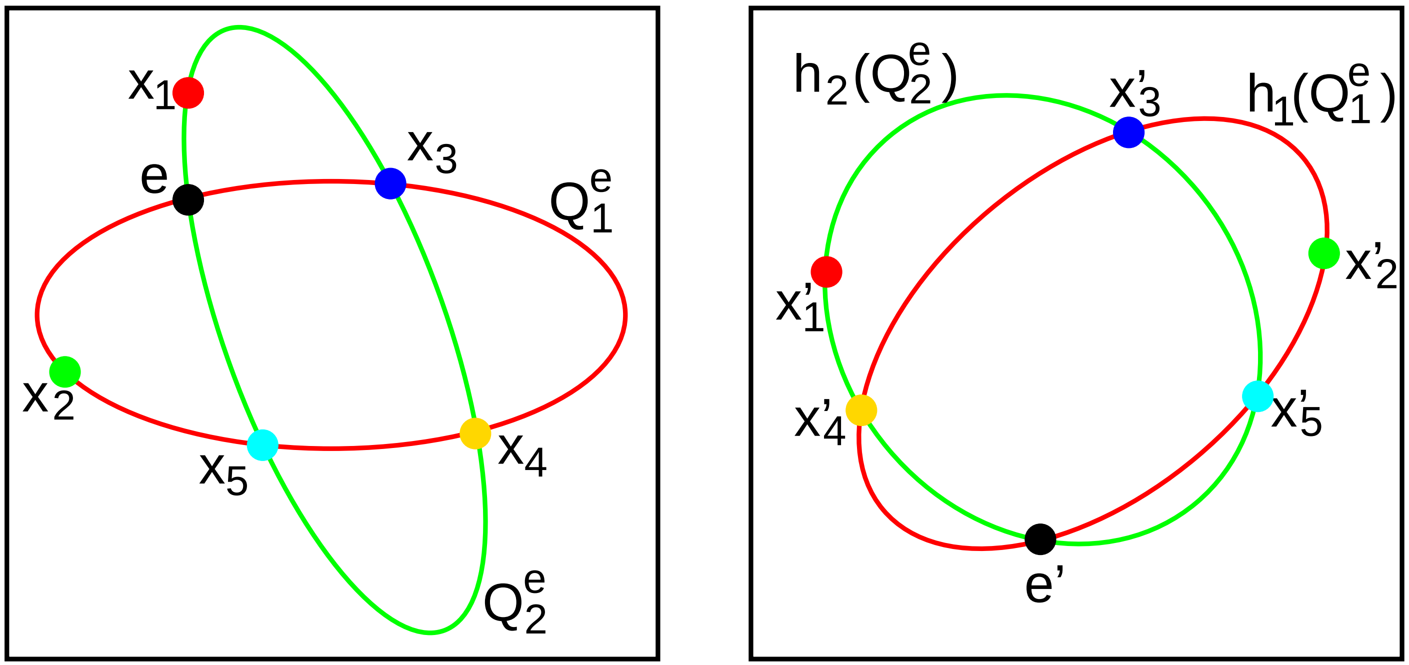


image 1 image 2

All five conics $h_1(Q_1^e), \ldots, h_5(Q_5^e)$ intersect in a single point e' $\Rightarrow$

e and e' are in functional relationship, $e' = f(e)$.

What if e is conconic with x_1, x_2, x_3, x_4, x_5?

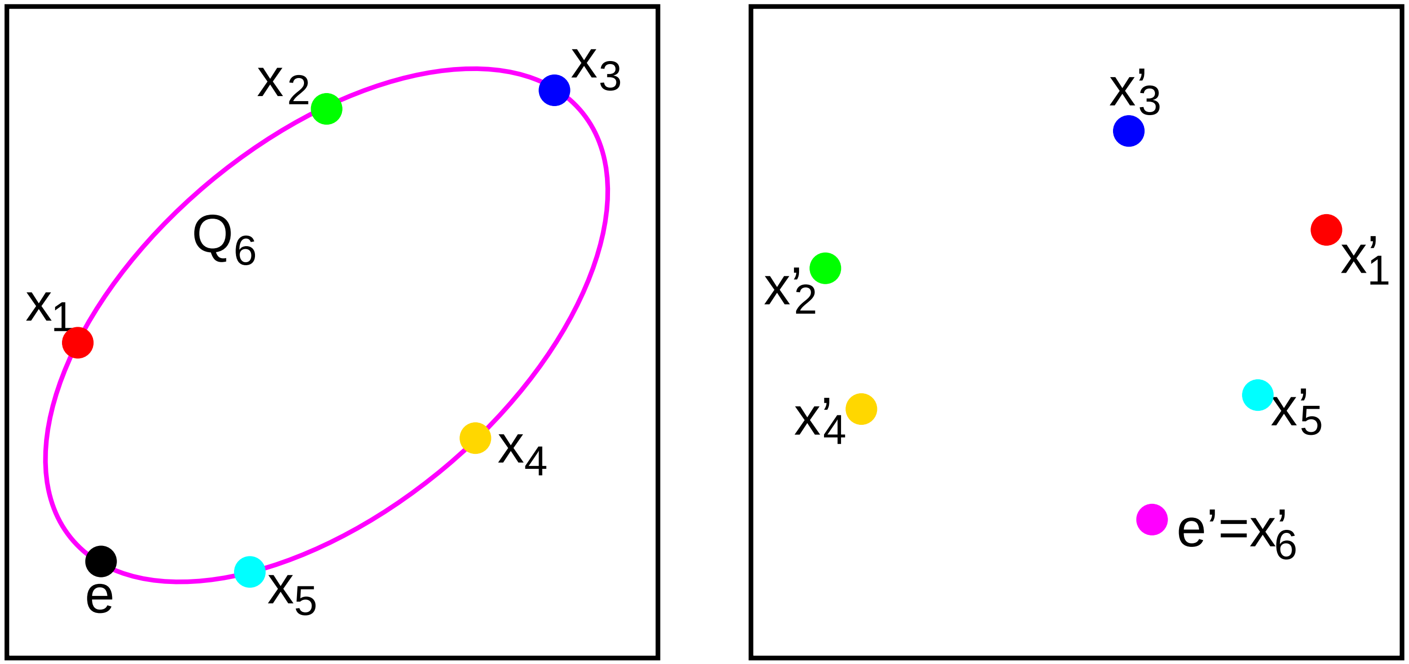


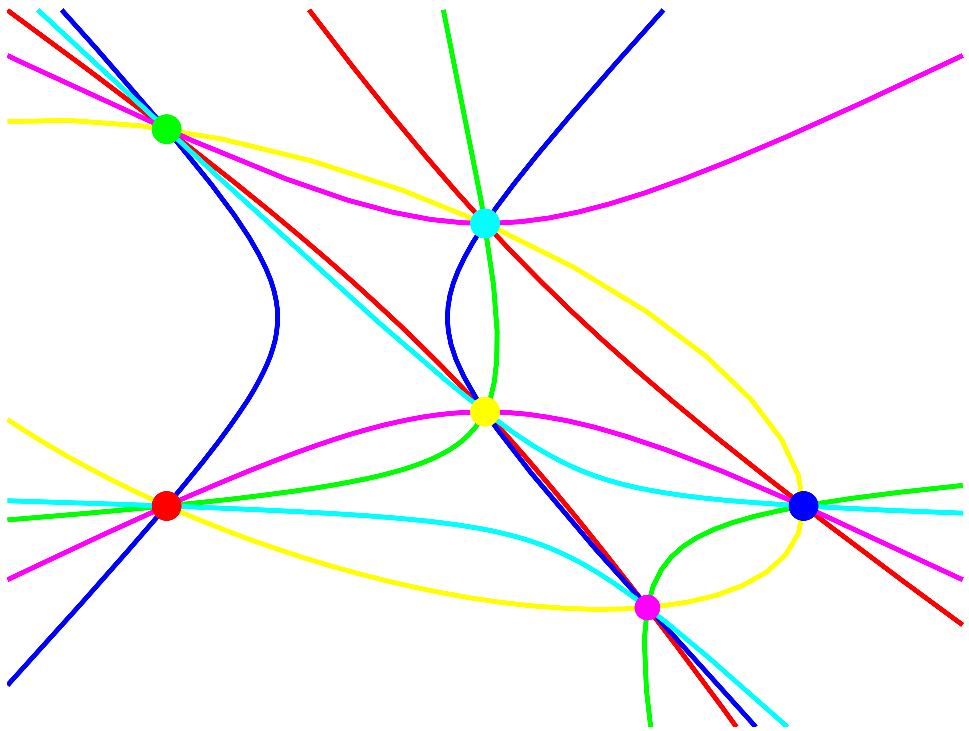
image 1 image 2

$$f[\text{conic}(x_1, \ldots, x_5)] = x'_6$$
$$f[\text{conic}(x'_1, \ldots, x'_5)] = x_6$$
$\Rightarrow$

Pairs $(x_1, x_1), \ldots, (x_5, x'_5)$ uniquely define a new point pair (x_6, x'_6).

$\{(x_1, x_1), \ldots, (x_6, x'_6)\}$ is a degenerate configuration of 6 correspondences.
All 6 pairs play the same rôle.

Denote: $Q_i =$ conic given by $\{x_1, \dots, x_6\} \setminus x_i$
$Q'_i =$ conic given by $\{x'_1, \dots, x'_6\} \setminus x'_i$



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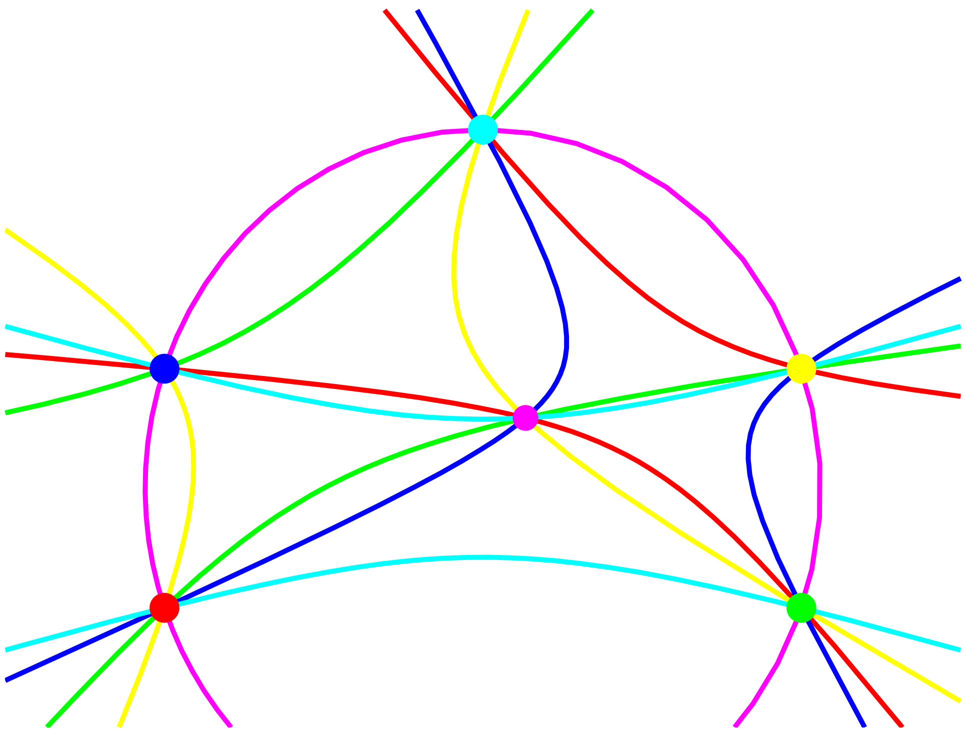

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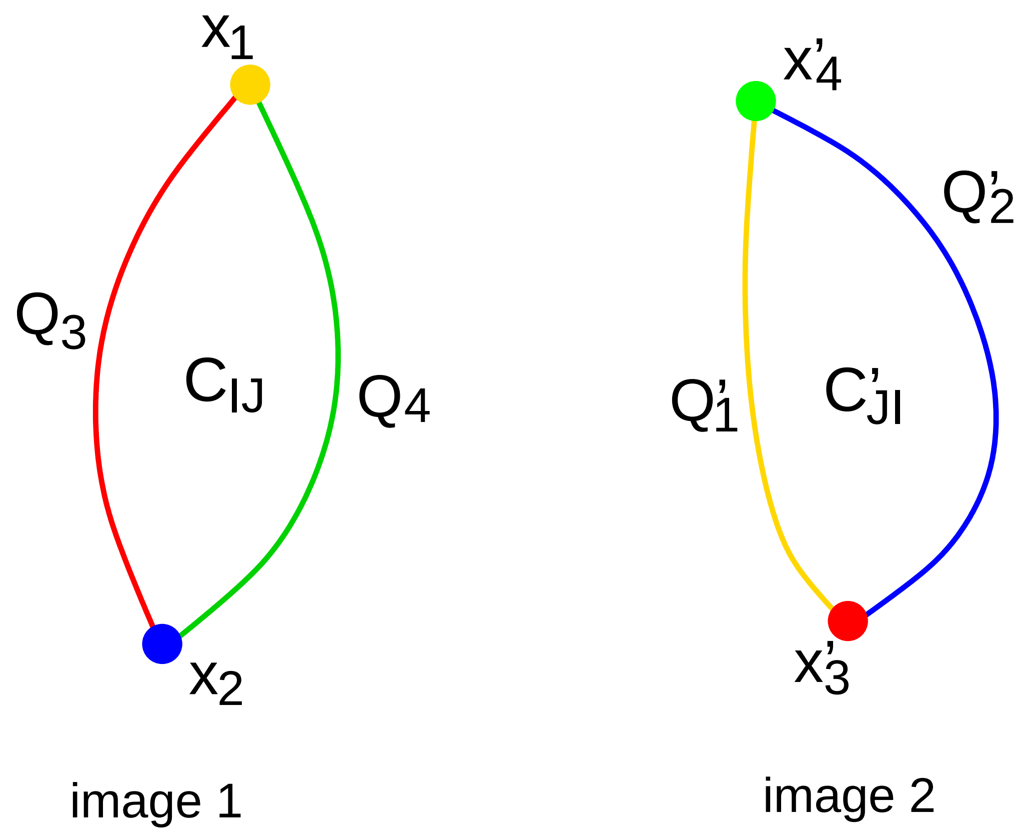
Classical result [Sturm-1908]:

- Function $e' = f(e)$ is a Cremona mapping of degree 5.
- x_i and x'_i are its exceptional points: $f(x_i)$ and $f^{-1}(x'_i)$ are undefined.
- Q_i and Q'_i are its exceptional curves: $f(Q_i) = x'_i$, $f(Q'_i) = x_i$.

m p
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

Conics $Q_1, \ldots, Q_6$ partition $\mathbb{P}^2$ into cells with piecewise conic boundaries:



C_{IJ} denotes the cell whose boundary is formed by

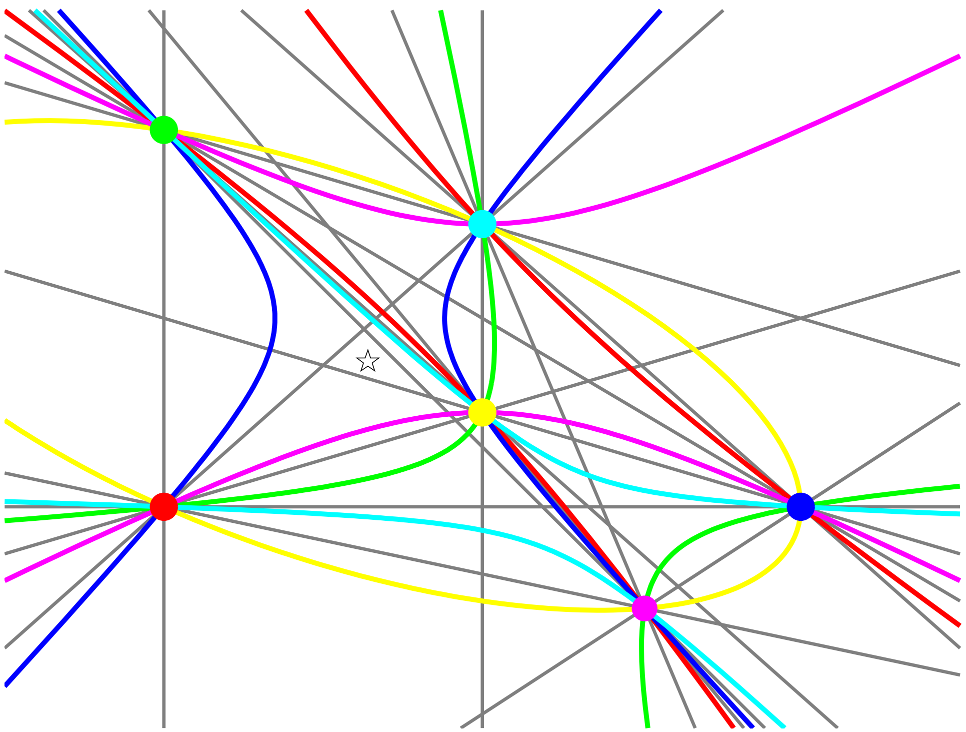
- conics with indices $I \subset \{1, \ldots, 6\}$ and
- points with indices $J \subset \{1, \ldots, 6\}$.

f induces correspondence on cells: $f(C_{IJ}) = C'_{JI}$

Recall: f satisfies

- $f(C_{IJ}) = C'_{JI}$
- $f(x_i x_j) = x'_i x'_j$.

Moreover, if e crosses line $x_i x_j$, then e' crosses line $x'_i x'_j$:



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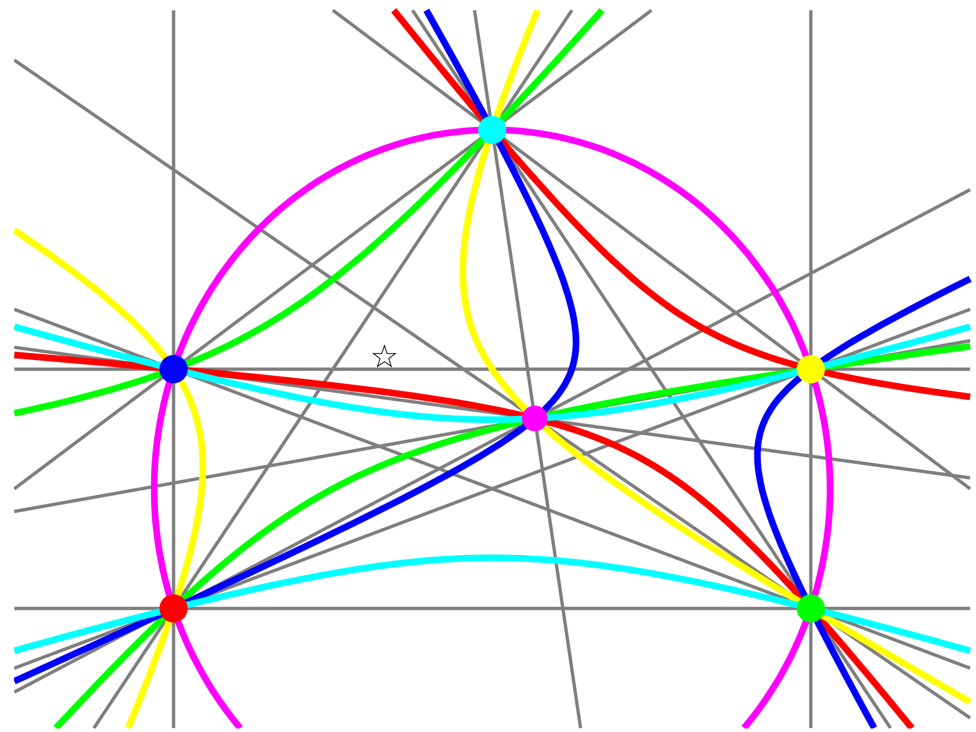

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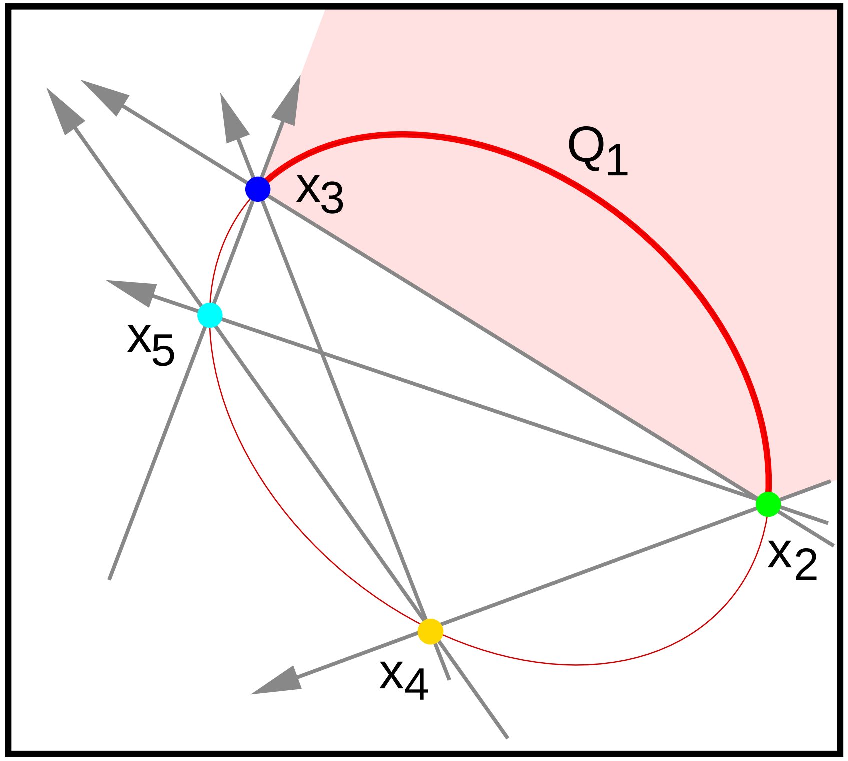
The property "*being an allowed epipole*" is constant within each piecewise-conic cell.

Trick: Consider $\boxed{e' = x'_1}$ (in a limit!). Then e must lie simultaneously on:

- Q_1
- the RHS of the $\binom{4}{2} = 6$ lines $x_i x_j$ where
  - $i, j = 2, \dots, 5$
  - orientation of line $x_i x_j$ is given by $-\mathrm{sign}|\mathbf{x}'_1, \mathbf{x}'_i, \mathbf{x}'_j|$.



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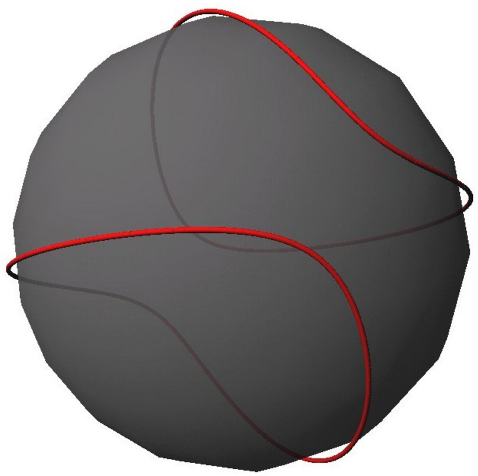
$\Rightarrow$ e lies on one of the four segments of Q_1 or does not exist.

continuity of f $\Rightarrow$ The allowed segments of conics $Q_1, \dots, Q_5$ uniquely delineate the allowed cells.

Example Sets of Allowed Segments/Cells

Conic in $\mathbb{S}^2$



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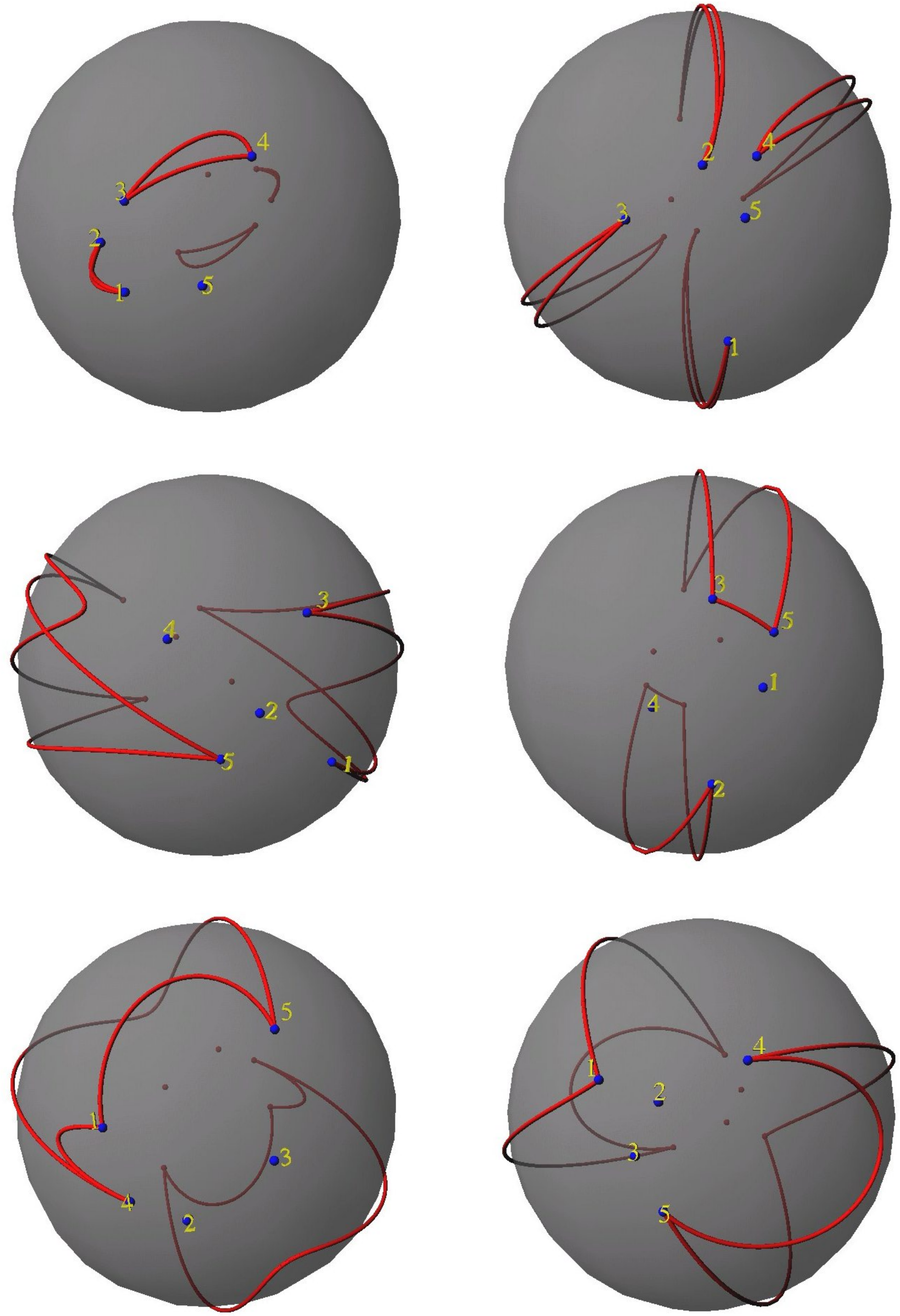

Given five points $x_1, \ldots, x_5$, find out where they are on the conic determined by them. Two parts:

- Projective part: 12 possible projective orders of 5 "beads on a ring:

$$\{(12345), \quad (12354), \quad (12534), \quad (15234), \\ (12435), \quad (12453), \quad (12543), \quad (15243), \\ (14235), \quad (14253), \quad (14523), \quad (15423)\}.$$

  Distinguished by separations $x_i(x_j, x_k) \,||\, x_i(x_l, x_m)$ for different i, j, k, l, k.

- Orientation part: use pivot line $l = (12 \vee 34)(14 \vee 23)$