Bounds on Weighted CSPs
Using Constraint Propagation and Super-Reparametrizations

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(Binary) Weighted Constraint Satisfaction Problem (WCSP)

- finite set of variables $V$
- finite domain $D$ of each variable
- set of variable pairs (edges) $E$
- find assignment $x \in D^V$ (i.e., $x : V \to D$) maximizing

$$F(x \mid f) = \sum_{i \in V} f_i(x_i) + \sum_{\{i,j\} \in E} f_{ij}(x_i, x_j)$$  \hspace{1cm} (1)

- weight functions $f_i : D \to \mathbb{R}$ and $f_{ij} : D^2 \to \mathbb{R}$
(Binary) Weighted Constraint Satisfaction Problem (WCSP)

- finite set of **variables** $V$
- finite **domain** $D$ of each variable
- set of variable pairs (edges) $E$
- find assignment $x \in D^V$ (i.e., $x : V \to D$) maximizing
  \[
  F(x \mid f) = \sum_{i \in V} f_i(x_i) + \sum_{\{i,j\} \in E} f_{ij}(x_i, x_j)
  \] (1)
- **weight functions** $f_i : D \to \mathbb{R}$ and $f_{ij} : D^2 \to \mathbb{R}$
- WCSP instances are identified with vectors $f \in \mathbb{R}^T$
- set of **tuples**:
  \[
  T = \{ (i, k) \mid i \in V, \; k \in D \} \cup \{ \{(i, k), (j, l)\} \mid \{i, j\} \in E, \; k, l \in D \}
  \] (2)
- Note: $F(x \mid f)$ is linear in $f$
(Binary) Constraint Satisfaction Problem (CSP)

- \( V, D, E, T \) as for WCSP, recall:

\[
T = \{(i, k) \mid i \in V, k \in D\} \cup \{(i, k), (j, l)\} \mid \{i, j\} \in E, k, l \in D\}
\]

- a set \( A \subseteq T \) of **allowed tuples** (tuples in \( T - A \) are forbidden)
(Binary) Constraint Satisfaction Problem (CSP)

- $V, D, E, T$ as for WCSP, recall:

$$T = \{(i, k) \mid i \in V, k \in D\} \cup \{(i, k), (j, l)\} \mid \{i, j\} \in E, k, l \in D$$

- A set $A \subseteq T$ of allowed tuples (tuples in $T - A$ are forbidden)

- CSP instances are identified with subsets of $T$

- Assignment $x \in D^V$ is a solution to CSP $A$ if

$$\forall i \in V : (i, x_i) \in A$$
$$\forall \{i, j\} \in E : \{(i, x_i), (j, x_j)\} \in A$$
For a WCSP \( f \in \mathbb{R}^T \):

- upper bound on the optimal value of WCSP \( f \):

\[
B(f) = \sum_{i \in V} \max_{k \in D} f_i(k) + \sum_{\{i, j\} \in E} \max_{k, l \in D} f_{ij}(k, l)
\]  

(3)
Upper Bound and Active-Tuple CSP

For a WCSP $f \in \mathbb{R}^T$:

- **upper bound** on the optimal value of WCSP $f$:

$$B(f) = \sum_{i \in V} \max_{k \in D} f_i(k) + \sum_{\{i,j\} \in E} \max_{k,l \in D} f_{ij}(k, l) \geq \max_{x \in D^V} \left( \sum_{i \in V} f_i(x_i) + \sum_{\{i,j\} \in E} f_{ij}(x_i, x_j) \right)$$  \hspace{1cm} (3)
Upper Bound and Active-Tuple CSP

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- tuple $t = (i, k) \in T$ is **active** if $f_i(k) = \max_{k' \in D} f_i(k')$

- tuple $t = \{(i, k), (j, l)\} \in T$ is **active** if $f_{ij}(k, l) = \max_{k', l' \in D} f_{ij}(k', l')$
For a WCSP $f \in \mathbb{R}^T$:

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- set of all **active tuples** for $f$ is denoted by $A^*(f) \subseteq T$
Upper Bound and Active-Tuple CSP

For a WCSP $f \in \mathbb{R}^T$:

- **upper bound** on the optimal value of WCSP $f$:

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**Theorem**\(^1\): Upper bound is tight (i.e., $B(f) = \max_x F(x \mid f)$) iff CSP $A^*(f)$ is satisfiable.

\(^1\)Werner: A linear programming approach to max-sum problem: A review
Example

\[ V = \{1, 2\}, \ D = \{A, B\}, \ E = \\{\{1, 2\}\}, \ F(x | f) = f_1(x_1) + f_2(x_2) + f_{12}(x_1, x_2) \]
WCSP $f$ is a **reparametrization** of WCSP $g$ if $F(x|f) = F(x|g)$ for all $x$. 

Given a WCSP $g \in \mathbb{R}^T$, minimize the upper bound over reparametrizations:

$$\min_{f \in \mathbb{R}^T} B(f) \quad \text{subject to} \quad F(x|f) = F(x|g) \quad \forall x \in \mathbb{D}$$

A polynomially-sized LP formulation exists:

2. dual of the basic LP relaxation of WCSP $g$

Many algorithms for its (approximate) optimization:

(a) block-coordinate descent (e.g., max-sum diffusion,

2. duality of the basic LP relaxation of WCSP $g$

Kovalevsky et al.: A diffusion algorithm for decreasing energy of max-sum labeling problem

4. Kolmogorov: Convergent tree-reweighted message passing for energy minimization

5. Tourani et al.: Taxonomy of dual block-coordinate ascent methods for discrete energy minimization

6. Larrosa et al.: Existential arc consistency: getting closer to full arc consistency in weighted CSPs

7. Koval et al.: Two-dimensional programming in image analysis problems

8. Cooper et al.: Soft arc consistency revisited
Minimizing the Upper Bound by Reparametrizations

WCSP $f$ is a **reparametrization** of WCSP $g$ if $F(x|f) = F(x|g)$ for all $x$.

Given a WCSP $g \in \mathbb{R}^T$, minimize the upper bound over reparametrizations:

$$\min_{f \in \mathbb{R}^T} B(f) \quad \text{subject to} \quad F(x|f) = F(x|g) \quad \forall x \in D^V \quad (4)$$

- a polynomially-sized LP formulation exists$^2$ – dual of the basic LP relaxation of WCSP $g$

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$^2$Werner: A linear programming approach to max-sum problem: A review
Minimizing the Upper Bound by Reparametrizations

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- a polynomially-sized LP formulation exists\(^2\) – dual of the basic LP relaxation of WCSP $g$
- many algorithms for its (approximate) optimization:
  - (a) block-coordinate descent (e.g., max-sum diffusion\(^2,3\), TRWS\(^4\), SPAM\(^5\), ...)
  - (b) soft local consistencies (e.g., EDAC\(^6\), AugDAG\(^7\)/VAC\(^8\), ...)

\(^2\)Werner: A linear programming approach to max-sum problem: A review
\(^3\)Kovalevsky et al.: A diffusion algorithm for decreasing energy of max-sum labeling problem
\(^4\)Kolmogorov: Convergent tree-reweighted message passing for energy minimization
\(^5\)Tourani et al.: Taxonomy of dual block-coordinate ascent methods for discrete energy minimization
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\(^7\)Koval et al.: Two-dimensional programming in image analysis problems
\(^8\)Cooper et al.: Soft arc consistency revisited
Minimizing the Upper Bound by Super-Reparametrizations

WCSP $f$ is a **super-reparametrization** of WCSP $g$ if $F(x | f) \geq F(x | g)$ for all $x$. 

Given a WCSP $g \in \mathcal{R}_T$, minimize the upper bound over super-reparametrizations:

$$\min_{f \in \mathcal{R}_T} B(f)$$

subject to

$$F(x | f) \geq F(x | g) \forall x \in \mathcal{D}_V$$

for any feasible $f$:

$$B(f) \geq \max_x F(x | f) \geq \max_x F(x | g)$$

optimal value of (5) is

$$\text{feasible } f \in \mathcal{R}_T \text{ is optimal for (5) iff CSP } A^*(f) \text{ has a solution } x \text{ with } F(x | f) = F(x | g)$$

satisfiability of $A^*(f)$ is a necessary (but generally insufficient) condition of optimality.
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$$

(5)

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Komodakis, et al.: Beyond loose LP-relaxations: Optimizing MRFs by repairing cycles
Minimizing the Upper Bound by Super-Reparametrizations

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Given a WCSP $g \in \mathbb{R}^T$, minimize the upper bound over super-reparametrizations:\footnote{Komodakis, et al.: Beyond loose LP-relaxations: Optimizing MRFs by repairing cycles}

$$\min_{f \in \mathbb{R}^T} B(f) \quad \text{subject to} \quad F(x | f) \geq F(x | g) \quad \forall x \in D^V$$

for any feasible $f$: $B(f) \geq \max_x F(x | f) \geq \max_x F(x | g)$
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- optimal value of (5) is $\max_x F(x | g)$

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9Komodakis, et al.: Beyond loose LP-relaxations: Optimizing MRFs by repairing cycles
WCSP $f$ is a super-reparametrization of WCSP $g$ if $F(x \mid f) \geq F(x \mid g)$ for all $x$.

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- for any feasible $f$: $B(f) \geq \max_x F(x \mid f) \geq \max_x F(x \mid g)$
- optimal value of (5) is $\max_x F(x \mid g)$
- feasible $f \in \mathbb{R}^T$ is optimal for (5) iff CSP $A^*(f)$ has a solution $x$ with $F(x \mid f) = F(x \mid g)$
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\[^9\text{Komodakis, et al.: Beyond loose LP-relaxations: Optimizing MRFs by repairing cycles}\]
Iterative Scheme

**Theorem:** Let $f \in \mathbb{R}^T$. CSP $A^*(f)$ is unsatisfiable iff $\exists h \in \mathbb{R}^T$ with $B(f + h) < B(f)$ and $F(x|h) \geq 0$ for all $x \in D^V$. 
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Iterative scheme for (approximate) optimization:

1. Initialize $f := g$.
2. If CSP $A^*(f)$ is satisfiable, stop.
3. Find certificate $h$.
4. Update $f := f + h$ and go to 2.

Properties:

- $B(f)$ decreases after each iteration
- $F(x| f + h) = F(x| f) + F(x| h) \geq F(x| f)$
- $\max_x F(x| f)$ increases or stays the same after each iteration

obtained bound is limited by the fact that $B(f) \geq \max_x F(x| f)$
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Iterative Scheme with Constraint Propagation

Idea: Try to detect unsatisfiability of CSP $A^*(f)$ by constraint propagation.

Next, we will show:

- how to compute a certificate $h$ using any constraint propagation algorithm
- how to obtain good certificates
- experiments with singleton arc consistency
Deactivating Directions

**Definition:** Let \( A \subseteq T \) and \( S \subseteq A, \ S \neq \emptyset \).

An **\( S \)-deactivating direction** for **CSP** \( A \) is a vector \( d \in \mathbb{R}^T \) satisfying

(a) \( d_t < 0 \) for all \( t \in S \),
(b) \( d_t = 0 \) for all \( t \in A - S \),
(c) \( F(x|d) \geq 0 \) for all \( x \in D^V \).
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An **$S$-deactivating direction for CSP $A$** is a vector $d \in \mathbb{R}^T$ satisfying

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**Theorem:** An $S$-deactivating direction $d \in \mathbb{R}^T$ for $A$ exists iff CSPs $A$ and $A - S$ have the same solution set.
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**Theorem:** An $S$-deactivating direction $d \in \mathbb{R}^T$ for $A$ exists iff CSPs $A$ and $A - S$ have the same solution set.

**Example:** $V = \{1, 2\}$, $D = \{A, B\}$, $E = \{\{(1, 2)\}\}$, $S = \{((1, A), (2, B))\}$
Deactivating Directions

Domain wipeout: for some $i \in V$, $(i, k) \notin A$ for all $k \in D$

Edge wipeout: for some $\{i, j\} \in E$, $\{(i, k), (j, l)\} \notin A$ for all $k, l \in D$

$\implies$ A is unsatisfiable
Deactivating Directions

**Domain wipeout:** for some $i \in V$, $(i, k) \notin A$ for all $k \in D$

**Edge wipeout:** for some $\{i, j\} \in E$, $\{(i, k), (j, l)\} \notin A$ for all $k, l \in D$

\[ \implies A \text{ is unsatisfiable} \]

**Theorem:** Let $f \in \mathbb{R}^T$ and $d$ be an $S$-deactivating direction for $A^*(f)$.

- If there is domain or edge wipeout in $A^*(f) - S$, then $\exists \alpha > 0: B(f + \alpha d) < B(f)$.
- Otherwise, $\exists \alpha > 0: B(f + \alpha d) = B(f)$ and $A^*(f + \alpha d) = A^*(f) - S$. 

Computing Deactivating Directions

One iteration of a local consistency algorithm applied to CSP $A \subseteq T$
- identify tuples $S \subseteq A$ such that CSPs $A$ and $A - S$ have the same solution set
- forbid these tuples: update CSP to $A - S$
Computing Deactivating Directions

One iteration of a local consistency algorithm applied to CSP $A \subseteq T$

- identify tuples $S \subseteq A$ such that CSPs $A$ and $A - S$ have the same solution set
- forbid these tuples: update CSP to $A - S$

Refinement: proof set $P \subseteq T - A$

- for any CSP $A' \subseteq T - P$, CSPs $A'$ and $A' - S$ have the same solution set
- e.g., $P = T - A$

Theorem: $S$-deactivating direction for $A$:

$$d_t = \begin{cases} 
-1 & \text{if } t \in S \\
0 & \text{if } t \in P \\
\text{otherwise} 
\end{cases}$$

where $n$ is the number of weight functions (unary or binary) with at least one tuple in $S$. Note: $P$ is preferred to be small (so that values $F(x|d)$ are small).
Computing Deactivating Directions

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- identify tuples $S \subseteq A$ such that CSPs $A$ and $A - S$ have the same solution set
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**Example:** $S = \{\{(1, A), (2, B)\}\}$, $P = \{(2, B)\}$, CSP $A$:

```
+---+---+
| A | B |
+---+---+
  |   |
+ ---+---+
```
Computing Deactivating Directions

One iteration of a local consistency algorithm applied to CSP $A \subseteq T$

- identify tuples $S \subseteq A$ such that CSPs $A$ and $A - S$ have the same solution set
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- for any CSP $A' \subseteq T - P$, CSPs $A'$ and $A' - S$ have the same solution set
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**Example** (Singleton Arc Consistency): Let $i \in V$, $k \in D$, and $A \subseteq T$.

If $A|_{x_i = k}$ has empty AC closure:

- There is no solution of CSP $A$ with $x_i = k$ and we can forbid $(i, k)$.
- $S = \{(i, k)\}$
- $P$ is, e.g., the set of forbidden tuples needed to infer empty AC closure of $A|_{x_i = k}$. 

\[d_t = \begin{cases} -1 & \text{if } t \in S \\ n & \text{if } t \in P \\ 0 & \text{otherwise} \end{cases}\] 

where $n$ is the number of weight functions (unary or binary) with at least one tuple in $S$. 

Note: $P$ is preferred to be small (so that values $F(x|d_t)$ are small).
Computing Deactivating Directions

One iteration of a local consistency algorithm applied to CSP $A \subseteq T$

- identify tuples $S \subseteq A$ such that CSPs $A$ and $A - S$ have the same solution set
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- for any CSP $A' \subseteq T - P$, CSPs $A'$ and $A' - S$ have the same solution set
- e.g., $P = T - A$

Theorem: $S$-deactivating direction for $A$:

$$d_t = \begin{cases} 
-1 & \text{if } t \in S \\
 n & \text{if } t \in P \\
 0 & \text{otherwise} 
\end{cases}$$

(6)

where $n$ is the number of weight functions (unary or binary) with at least one tuple in $S$.

Note: $P$ is preferred to be small (so that values $F(x|d)$ are small)
Composing Deactivating Directions

Let $f \in \mathbb{R}^T$.

Given CSP $A_0 = A^*(f)$, apply constraint propagation to forbid some tuples:

- Forbid tuples $S_0 \subseteq A_0$, let $A_1 = A_0 - S_0$, store $S_0$-deactivating direction $d^0$ for $A_0$. 

... 

- Forbid tuples $S_q \subseteq A_q$, let $A_{q+1} = A_q - S_q$, store $S_q$-deactivating direction $d^q$ for $A_q$.

Theorem: The sequence $d^0, d^1, ..., d^q$ can be composed into a single $S$-deactivating direction $d$ for $A^*(f)$ where $S = S_0 \cup S_1 \cup ... \cup S_q$.

Note: $A_{q+1} = A_0 - S$.

Corollary: If there is domain or edge wipeout in $A_{q+1} = A^*(f) - S$, $d$ can be used to improve the bound as $B(f^\alpha d) < B(f)$.
Composing Deactivating Directions

Let $f \in \mathbb{R}^T$.

Given CSP $A_0 = A^*(f)$, apply constraint propagation to forbid some tuples:

- Forbid tuples $S_0 \subseteq A_0$, let $A_1 = A_0 - S_0$, store $S_0$-deactivating direction $d^0$ for $A_0$.
- Forbid tuples $S_1 \subseteq A_1$, let $A_2 = A_1 - S_1$, store $S_1$-deactivating direction $d^1$ for $A_1$.

...
Composing Deactivating Directions

Let \( f \in \mathbb{R}^T \).

Given CSP \( A_0 = A^*(f) \), apply constraint propagation to forbid some tuples:

- Forbid tuples \( S_0 \subseteq A_0 \), let \( A_1 = A_0 - S_0 \), store \( S_0 \)-deactivating direction \( d^0 \) for \( A_0 \).
- Forbid tuples \( S_1 \subseteq A_1 \), let \( A_2 = A_1 - S_1 \), store \( S_1 \)-deactivating direction \( d^1 \) for \( A_1 \).
  
  :  
  
- Forbid tuples \( S_q \subseteq A_q \), let \( A_{q+1} = A_q - S_q \), store \( S_q \)-deactivating direction \( d^q \) for \( A_q \).

Theorem: The sequence \( d^0, d^1, \ldots, d^q \) can be composed into a single \( S \)-deactivating direction \( d \) for \( A^*(f) \) where \( S = S_0 \cup S_1 \cup \ldots \cup S_q \).

Note: \( A_{q+1} = A_0 - S \)

Corollary: If there is domain or edge wipeout in \( A_{q+1} = A^*(f) - S \), \( d \) can be used to improve the bound as \( B(f + \alpha d) < B(f) \).
Composing Deactivating Directions

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Given CSP $A_0 = A^*(f)$, apply constraint propagation to forbid some tuples:
- Forbid tuples $S_0 \subseteq A_0$, let $A_1 = A_0 - S_0$, store $S_0$-deactivating direction $d^0$ for $A_0$.
- Forbid tuples $S_1 \subseteq A_1$, let $A_2 = A_1 - S_1$, store $S_1$-deactivating direction $d^1$ for $A_1$.
- Forbid tuples $S_q \subseteq A_q$, let $A_{q+1} = A_q - S_q$, store $S_q$-deactivating direction $d^q$ for $A_q$.

**Theorem:** The sequence $d^0, d^1, \ldots, d^q$ can be composed into a single $S$-deactivating direction $d$ for $A^*(f)$ where $S = S_0 \cup S_1 \cup \ldots \cup S_q$.

Note: $A_{q+1} = A_0 - S$

**Corollary:** If there is domain or edge wipeout in $A_{q+1} = A^*(f) - S$, $d$ can be used to improve the bound as $B(f + \alpha d) < B(f)$. 

Iterative scheme for computing an upper bound $B(f)$ on $\max_x F(x \mid g)$:

1: Initialize $f := g$.

2: Apply constraint propagation on $A^*(f)$ while storing deactivating directions $d^0, ..., d^q$.

3: If there is domain or edge wipeout:
   3.1: Compose (possibly a subset of) the sequence $d^0, ..., d^q$ into a single vector $d$.
   3.2: Compute step size $\alpha$.
   3.3: Update $f := f + \alpha d$, go to 2.

4: Return $B(f)$. 
Overview

Iterative scheme for computing an upper bound $B(f)$ on $\max_x F(x | g)$:

1: Initialize $f := g$.
2: Apply constraint propagation on $A^*(f)$ while storing deactivating directions $d^0, ..., d^q$
3: If there is domain or edge wipeout:
   3.1: Compose (possibly a subset of) the sequence $d^0, ..., d^q$ into a single vector $d$.
   3.2: Compute step size $\alpha$.
   3.3: Update $f := f + \alpha d$, go to 2.
4: Return $B(f)$.
Experiments

Data: Cost Function Library benchmark\textsuperscript{10}

Compared methods:
- Virtual singleton arc consistency via super-reparametrizations (VSAC-SR)
- Virtual cycle consistency via super-reparametrizations (VCC-SR) \textsuperscript{(11)}
- EDAC\textsuperscript{12}, VAC\textsuperscript{13}, pseudo-triangles\textsuperscript{14}, triangle-based consistencies: PIC, EDPIC, maxRPC, EDmaxRPC\textsuperscript{15}

Only the upper bound is computed.

\textsuperscript{10}https://forgemia.inra.fr/thomas.schiex/cost-function-library
\textsuperscript{11}Komodakis, et al.: Beyond loose LP-relaxations: Optimizing MRFs by repairing cycles
\textsuperscript{12}Larrosa et al.: Existential arc consistency: getting closer to full arc consistency in weighted CSPs
\textsuperscript{13}Cooper et al.: Soft arc consistency revisited
\textsuperscript{14}Option \texttt{−t = 8000} in toulbar2; https://miat.inrae.fr/toulbar2
\textsuperscript{15}Nguyen, et al.: Triangle-based consistencies for cost function networks
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</table>

Average over all groups: 5371 | 0.55 | 13.17 | 495.38 | 417.59 | 143.17 | 471.21 | 471.49 | 491.88 | 538.35
Average over groups with ≥ 5 instances: 5369 | 0.58 | 14.04 | 527.54 | 445.20 | 112.82 | 498.80 | 499.08 | 517.49 | 566.88