Why quantum logic cannot be classical

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Classical event structure

$\sigma$-algebra of sets, $\mathcal{L} \subseteq 2^U$:

- $U \in \mathcal{L}$
- $\{A_i \mid i \in \mathbb{N}\} \subseteq \mathcal{L} \implies \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{L}$

State ($=\text{probability measure}$) $s: \mathcal{L} \to [0,1]$:

- $s(U) = 1$
- $\{A_i \mid i \in \mathbb{N}\} \subseteq \mathcal{L}, A_i \cap A_j = \emptyset$ for $i \neq j \implies s \left( \bigcup_{i \in \mathbb{N}} A_i \right) = \sum_{i \in \mathbb{N}} s(A_i)$

$S(\mathcal{L}) := \text{state space}$ of $\mathcal{L}$; it is a Choquet simplex

Pure states: extreme points of $S(\mathcal{L})$

Two-valued states: $S(\mathcal{L}) \cap \{0,1\}^\mathcal{L}$

For $\sigma$-algebras:

- pure states = two-valued states = points in the Stone space
- state space (even the space of two-valued states) determines the whole structure

We need disjoint, not all unions!

$\sigma$-class of sets, $\mathcal{L} \subseteq 2^U$:

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Example 1

\(W/L = \text{Wins}/\text{Loses with/without player JJ}\)

\[
\begin{array}{c|c}
W & W \\
\hline
L & W \\
\hline
W & L \\
\hline
L & L \\
\end{array}
\]

\(b\)

\(\times\)

\(A\)

\[U = \{W|W, W|L, L|W, L|L\}\]

\[A = \{\emptyset, \{W|W, W|L\}, \{L|W, L|L\}, U\}\]

\[B = \{\emptyset, \{W|W, L|W\}, \{W|L, L|L\}, U\}\]

\[\mathcal{L} = A \cup B = \{\emptyset, a, a', b, b', U\}\]

\(\mathcal{L}\) is a (nondistributive modular) lattice called \(MO2\)
Pure states:
\[ s : \mathcal{L} \rightarrow \{0, 1\} \]

<table>
<thead>
<tr>
<th></th>
<th>(s(A))</th>
<th>(s(B))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
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<td>0</td>
<td>0</td>
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<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

(S0) \[ s(U) = 1 \]
(S1) \[ s(x') = 1 - s(x) \]

All states:
\[ s : \mathcal{L} \rightarrow [0, 1], \text{ satisfy (S0), (S1)} \]
\[ s(A) = p, \ s(B) = q, \ p, q \in [0, 1] \text{ arbitrary} \]
Example 2

Example 1 with one more result, \( c = \text{match cancelled} \)
\[ A = \{0, a, c, (a \lor c)', a \lor c, a', c', 1\} \]
\[ B = \{0, b, c, (b \lor c)', b \lor c, b', c', 1\} \]
\[ A \cap B = \{0, c, c', 1\} \]
\[ \mathcal{L} = A \cup B = \{0, a, b, c, a \lor c, b \lor c, (a \lor c)', (b \lor c)', a', b', c', 1\} \]
Pure states:

<table>
<thead>
<tr>
<th>$s(A)$</th>
<th>$s(B)$</th>
<th>$s(C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
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<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

All states:
$s(A) = p$, $s(B) = q$, $s(C) = r$, $r \in [0, 1]$ arbitrary, $p, q \in [0, 1 - r]$
Example 3:

\[ \mathcal{K} = \{0, B, C, D, B', C', D', 1\}, \text{ where } D \text{ means "the fire-fly is not observed from } K" \]
\[ \mathcal{M} = \{0, A, C, E, A', C', E', 1\}, \text{ where } A \text{ means "the fire-fly is observed in the upper part"} \]
\[ \mathcal{J} = \{0, A, B, F, A', B', F', 1\} \]
\[ \mathcal{K} \cup \mathcal{M} \cup \mathcal{J} = \{0, A, B, C, D, E, F, A', B', C', D', E', F', 1\} \]

This is \textbf{not} a lattice.
Pure states:

<table>
<thead>
<tr>
<th>$s(A)$</th>
<th>$s(B)$</th>
<th>$s(C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
</tbody>
</table>

All states:

$s(A) = p$, $s(B) = q$, $s(C) = r$, \( p, q, r \in [0, 1] \), \( p + q \leq 1 \), \( p + r \leq 1 \), \( q + r \leq 1 \)
Example 3 (non-transparent barriers)

\[
\begin{array}{cc}
D & \times \\
\times & C \\
\times & B \\
\end{array}
\]

\[
\begin{array}{cc}
d & a \\
c & b \\
\end{array}
\times A
\]
Example 3 (non-transparent barriers)

Pure states:

<table>
<thead>
<tr>
<th></th>
<th>s(a)</th>
<th>s(b)</th>
<th>s(c)</th>
<th>s(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
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<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>5</td>
<td>0</td>
<td>0</td>
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<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<tr>
<td>7</td>
<td>0</td>
<td>0</td>
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**Orthomodular lattices**

*Definition:* An orthomodular lattice is a lattice with bounds 0, 1 equipped with a unary operation $'$ : $\mathcal{L} \rightarrow \mathcal{L}$ (orthocomplementation) such that, for all $a, b \in \mathcal{L}$,

- $a'' = a$
- $a \leq b \implies b' \leq a'$
- $a \land a' = 0$
- $a \leq b \implies b = a \lor (a' \land b)$ (orthomodular law)

**Orthogonality:** $a \perp b \iff a \leq b'$

(This condition is strictly stronger than the usual $a \land b = 0$.)

*Example:* A $\sigma$-class of subsets needs not be a lattice; if it is, it is an OML.
Structure of orthomodular lattices

Boolean subalgebra: \( \mathcal{M} \subseteq \mathcal{L} \) such that

- \( 0, 1 \in \mathcal{M} \),
- \( a \in \mathcal{M} \implies a' \in \mathcal{M} \),
- \( (\mathcal{M}, \leq \mathcal{M}, ' \uparrow \mathcal{M}) \) is a Boolean algebra.

Compatibility: \( a \leftrightarrow b \iff \exists \) Boolean subalgebra \( \mathcal{M} \): \( a, b \in \mathcal{M} \)

Block: a maximal Boolean subalgebra

Center: The set of all \( a \in \mathcal{L} \) such that \( \forall b \in \mathcal{L} : a \leftrightarrow b \)
= the set of all “absolutely compatible” elements
= the classical part of the system
= the intersection of all blocks

Atom: \( a \in \mathcal{L} \setminus \{0\} \) such that there is no \( b \) satisfying \( 0 < b < a \)
\( \mathcal{A}(\mathcal{L}) := \) the set of all atoms of \( \mathcal{L} \)

(\( \sigma \)-additivite) state: \( s: \mathcal{L} \to [0, 1] \) such that

- \( s(1) = 1 \)
- \( \{a_i \mid i \in \mathbb{N}\} \subseteq \mathcal{L}, a_i \perp a_j \text{ for } i \neq j \implies s(\bigvee_{i \in \mathbb{N}} a_i) = \sum_{i \in \mathbb{N}} s(a_i) \)
Orthomodular lattices as families of Boolean algebras

Every OML is the union of its maximal Boolean subalgebras (=blocks)

**Hypergraph**: a nonempty set (of *vertices*) and its covering by nonempty subsets (edges)

**Greechie diagram**: hypergraph whose vertices are atoms and edges are blocks

**State on a hypergraph**: evaluation of vertices such that the sum over each edge is 1

**Problem**: Which hypergraphs are Greechie diagrams of OMLs?

\[
\begin{align*}
K & \quad c & \quad d \\
M & \quad c & \quad d
\end{align*}
\]

\[
\begin{align*}
a & = (c \lor d)' = b \\
a \lor e & = b \lor f
\end{align*}
\]
\[
a \lor b = e \lor f \perp g \lor h
\]
Orthoalgebras

Allowed

Forbidden

\[
\begin{align*}
\text{Allowed} & \quad \text{Forbidden} \\
\begin{tikzpicture}
\node (a) at (0,0) [circle,fill,inner sep=1pt] {};
\node (b) at (1,0) [circle,fill,inner sep=1pt] {};
\node (c) at (0.5,1.5) [circle,fill,inner sep=1pt] {};
\draw (a) -- (b) -- (c) -- (a);
\end{tikzpicture}
\end{align*}
\]
In particular, the state space may be empty [Rogalewicz]:

---

[Diagram showing a state space structure with hexagonal and square patterns]
Smaller example with empty state space [Greechie]:

![Diagram](image-url)
Even smaller example with empty state space [R. Mayet]:

This is the smallest example with empty state space obtained by this technique and it is not unique [MN 08]; it has 19 blocks.

OMLs with \( \leq 5 \) blocks admit states [Riečanová 07].
Bell inequalities

\[ s(a) + s(b) - s(a \land b) \leq 1 \]
\[ 0 \geq s(a \land b) + s(b \land c) + s(c \land d) - s(a \land d) - s(b) - s(c) \]
\[ s(a) + s(b) + s(c) - s(a \land b) - s(a \land c) - s(b \land c) \leq 1 \]
\[ s(a \land b) + s(b \land c) + s(c \land d) - s(a \land d) - s(b) - s(c) \geq -1 \]

The first is equivalent to the \textit{valuation property}:
\[ s(a \land b) + s(a \lor b) = s(a) + s(b) \]

If the OML is not a Boolean algebra and admits a rich set of states, all Bell inequalities are violated.
Crucial example of a quantum structure: Hilbert lattice

$H$ ... a separable Hilbert space (real or complex)
$L(H)$ ... the set of all closed subspaces of $H$ (equivalently, all projectors of $H$)

\[
\begin{align*}
0 &= \{0\}, \\
1 &= H, \\
A \leq B &\iff A \subseteq B, \\
A \land B &= A \cap B, \\
A' &= \{x \in H \mid \forall y \in A : y \perp x\}, \\
A \lor B &= \text{Lin}(A \cup B),
\end{align*}
\]

where $\text{Lin}$ denotes the closed linear hull
States on Hilbert lattices in $\mathbb{R}^2$

The only restriction of states for $\dim P = 1$: \[ s(P') = 1 - s(P) \]

Many two-valued states = colourings of non-zero vectors by two colors (blue, red) such that each orthogonal basis contains exactly one red vector.
1. For $q \in H$, $\|q\| = 1$, define a vector state

$$s_q(Lin(\{y_1, \ldots, y_n\})) = \sum_{i=1}^{n} (q \cdot y_i)^2 = \sum_{i=1}^{n} \cos^2 \angle(q, y_i)$$

for any orthonormal basis $(y_1, \ldots, y_n)$ of $H$
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\]

for any orthonormal basis \((y_1, \ldots, y_n)\) of \( H \)

**Corollary:** \( s_q(q) = 1 \)
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**Corollary:** $s_q(q) = 1$

2. **Mixture** of vector states

$$s(P) = \sum_i c_i s_{q_i}(P),$$

where $c_i > 0$, $\sum_i c_i = 1$. 
States on Hilbert lattice $\mathbb{R}^3$

1. For $q \in H$, $||q|| = 1$, define a **vector state**

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**Corollary:** $s_q(q) = 1$

2. **Mixture** of vector states

$$s(P) = \sum_i c_i s_{q_i}(P), \text{ where } c_i > 0, \sum_i c_i = 1.$$  

3. What else?
States on Hilbert lattice $\mathbb{R}^3$

1. For $q \in H$, $||q|| = 1$, define a **vector state**

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**Corollary:** $s_q(q) = 1$

2. **Mixture** of vector states

$$s(P) = \sum_i c_i s_{q_i}(P), \text{ where } c_i > 0, \sum_i c_i = 1.$$

3. What else?

**Nothing!**

**Gleason’s Theorem** [Gleason 57]: For $\dim H \geq 3$, all states are mixtures of vector states.
Gleason’s Theorem

**Crucial case:** $H = \mathbb{R}^3$ (simplified proof by [Cooke, Keane, Moran 85]).

**Corollary 1:** The restriction of a state to 1D subspaces is continuous (proved by [von Neumann 1932] even for $\mathbb{R}^2$, error found by [Hermann 1935],

**Corollary 2:** A finitely-valued state is constant on 1D subspaces, i.e., it is a dimension function.
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There is no colouring of non-zero vectors by two colors (blue, red) such that each orthogonal basis contains exactly one red vector
Constructions proving Geometrical Lemmas

BGL

MGL
Constructions proving the non-existence of two-valued states in $\mathbb{R}^3$

It is possible to find a finite set of vectors whose orthogonality relations exclude the possibility of a two-valued state.

The smallest example known uses 31 vectors, the following uses 33 vectors:
Constructions proving the non-existence of two-valued states in $\mathbb{R}^4$

**Theorem**: [Cabello] There is no two-valued state on $\mathcal{L}(\mathbb{R}^4)$.

Take 36 vectors in $\mathbb{R}^4$ ($\bar{1}$ denotes $-1$):

<table>
<thead>
<tr>
<th>1000 1000 0100</th>
<th>1111 1111 1111 1111 1111 1111</th>
</tr>
</thead>
<tbody>
<tr>
<td>0100 0010 0010</td>
<td>1111 1111 1111 1111 1111 1111</td>
</tr>
<tr>
<td>0011 0101 1001</td>
<td>1100 1010 1100 1001 1010 1001</td>
</tr>
<tr>
<td>0011 0101 1001</td>
<td>0011 0101 0011 0110 0101 0110</td>
</tr>
</tbody>
</table>

Each of the 9 columns represents an orthogonal basis of $\mathbb{R}^4$ and each vector occurs twice. The number of vectors of unit state in this table must be both even and odd (9)—a contradiction.
Corollaries for group-valued states

**Theorem**: There is no nontrivial $\mathbb{Z}_2$-valued state, $s$, on $\mathcal{L}(R^4)$ which satisfies $s(1) = 1$.

**Theorem**: If $n \geq 4$, then there is no nontrivial $\mathbb{Z}_2$-valued state on $\mathcal{L}(R^n)$.

For $n \geq 5$ it follows from the above construction. The refinement for $n = 4$ is due to [Harding, Jager, Smith]

**Open problem**: Are there nontrivial $\mathbb{Z}_2$-valued states on $\mathcal{L}(R^3)$?
References


References


Details of pictures