Measures on tribes of fuzzy sets

Giuseppina Barbieri, Mirko Navara, Hans Weber

Dipartimento di Matematica e Informatica
Università degli Studi di Udine
Udine 33100, Italy
barbieri,weber@dimi.uniud.it

Center for Machine Perception
Faculty of Electrical Engineering
Czech Technical University
166 27 Praha, Czech Republic
navara@cmp.felk.cvut.cz
http://cmp.felk.cvut.cz/˜navara

Basic notions

classical probability theory	**fuzzy** probability theory
σ-algebra $\mathcal{T} \subseteq \mathbf{2}^X$	**tribe** $(\mathcal{T}, \dot\wedge)$, where $\mathcal{T} \subseteq [\mathbf{0}, \mathbf{1}]^X$ $\dot\wedge \colon [0,1]^2 \to [0,1]$ is a **t-norm**, i.e., commutative, associative, nondecreasing, and $a \dot\wedge 1 = a$
$\emptyset \in \mathcal{T}$ $A \in \mathcal{T} \Rightarrow A' := \mathbf{X} \setminus A \in \mathcal{T}$ $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$ $(A_n)_{n \in \mathbf{N}} \subseteq \mathcal{T}, A_n \nearrow A \Rightarrow A \in \mathcal{T}$	$0 \in \mathcal{T}$ $A \in \mathcal{T} \Rightarrow A' := \mathbf{1} - A \in \mathcal{T}$ $A, B \in \mathcal{T} \Rightarrow A \dot\cap B \in \mathcal{T}$ * $(A_n)_{n \in \mathbf{N}} \subseteq \mathcal{T}, A_n \nearrow A \Rightarrow A \in \mathcal{T}$
σ-additive measure $m \colon \mathcal{T} \to [0, \infty)$ $m(\emptyset) = 0$ $m(A \cup B)$ $= m(A) + m(B) - m(A \cap B)$ $A_n \nearrow A \Rightarrow m(A_n) \to m(A)$	**measure** ($\mathcal{T}$-measure) $m \colon \mathcal{T} \to [0, \infty)$ $m(0) = 0$ $m(A \dot\cup B)$ $= m(A) + m(B) - m(A \dot\cap B)$ ** $A_n \nearrow A \Rightarrow m(A_n) \to m(A)$

* $(A \dot\cap B)(x) := A(x) \dot\wedge B(x)$

** $\dot\cup$ is dual to $\dot\cap$, $A \dot\cup B := (A' \dot\cap B')'$

Underlying σ-algebra

(**center, boolean part, sharp elements**)

Proposition [Butnariu & Klement]:
Crisp elements of a tribe $(\mathcal{T}, \wedge)$, i.e., elements of $\mathcal{T} \cap \{0,1\}^X$, determine **always** a σ-algebra $\mathcal{B}(\mathcal{T})$ of subsets of X.

Frank t-norms:

$$a \underset{\mathbf{F_s}}{\wedge} b := \log_s \left(1 + \frac{(s^a - 1)(s^b - 1)}{s - 1} \right)$$

for $s \in (0, \infty) \setminus \{1\}$

limit cases:

$s \to 0 \quad \Rightarrow \quad a \underset{\mathbf{F_0}}{\wedge} b := \min(a, b)$ — **standard t-norm**

$s \to 1 \quad \Rightarrow \quad a \underset{\mathbf{F_1}}{\wedge} b := a \cdot b$ — **product t-norm**

$s \to \infty \quad \Rightarrow \quad a \underset{\mathbf{F_\infty}}{\wedge} b = a \underset{\mathbf{L}}{\wedge} b := \max(a + b - 1, 0)$ — **Łukasiewicz t-norm**

a t-norm $\underset{\cdot}{\wedge}$ is **strict** if it is continuous and $a < b,\ 0 < c \Rightarrow a \underset{\cdot}{\wedge} c < b \underset{\cdot}{\wedge} c$

a Frank t-norm $a \underset{\mathbf{F_s}}{\wedge} b$ is strict iff $s \in (0, \infty)$

Case 1: Łukasiewicz t-norm (MV-algebraic case)

Loomis–Sikorski Theorem for MV-algebras [Mundici, Dvurečenskij]:
Every σ-complete MV-algebra is an epimorphic image of a tribe $(\mathcal{T}, \underset{\mathbf{L}}{\wedge})$.

Theorem [Butnariu & Klement]: All elements of a tribe $(\mathcal{T}, \underset{\mathbf{L}}{\wedge})$ are $\mathcal{B}(\mathcal{T})$-measurable and every measure m on $(\mathcal{T}, \underset{\mathbf{L}}{\wedge})$ is a **linear integral measure**, i.e., it is of the form

$$\mathbf{m}(\mathbf{A}) = \int \mathbf{A}\,\mathbf{d}\boldsymbol{\mu}$$

where μ is a (classical) measure on $\mathcal{B}(\mathcal{T})$.

In this case, every measure is σ-additive in the usual sense.

The structure of $(\mathcal{T}, \underset{\mathbf{L}}{\wedge})$ may be complicated (characterized by [Klement, DN, MN]).

Case 1: Łukasiewicz t-norm II – characterization of tribes

$\frac{k}{n}$ generates

$$\mathbf{S_n} := \{0, \tfrac{1}{n}, \tfrac{2}{n}, \ldots, 1\} \subseteq [0,1]$$

in particular,

$$\mathbf{S_1} := \{0, 1\}$$

Theorem: For every tribe $(\mathcal{T}, \underset{\mathbf{L}}{\wedge})$ there are σ-filters ∇_n in $\mathcal{B}(\mathcal{T})$, $n \in N$, such that

$\nabla_m \subseteq \nabla_n$ whenever n is a divisor of m,

$\mathcal{T} = \{A \in [0,1]^X : \quad A$ is $\mathcal{B}(\mathcal{T})$-measurable, $\quad (\forall n \in N : A^{-1}(S_n) \in \nabla_n)\}$

Case 2: Frank t-norms

Theorem [Butnariu & Klement; MN; Barbieri & H. Weber]:
Let $(\mathcal{T}, \underset{\mathbf{F_s}}{\wedge})$ be a tribe, where $\underset{\mathbf{F_s}}{\wedge}$ is a strict Frank t-norm. Then all elements of $\mathcal{T}$ are $\mathcal{B}(\mathcal{T})$-measurable and every measure m on $(\mathcal{T}, \underset{\mathbf{F_s}}{\wedge})$ is of the form

$$\mathbf{m(A)} = \boldsymbol{\mu}(\textbf{Supp } \mathbf{A}) + \int \mathbf{A}\, \mathbf{d}\boldsymbol{\nu}$$

where **Supp** $A := \{x \in X : A(x) > 0\}$ and
μ, ν are (classical) measures on $\mathcal{B}(\mathcal{T})$.

$\mu(\textbf{Supp } A)$ ... **support measure**

Example [Barbieri & H. Weber]: A measure need not be monotonic, e.g.,

$$m(A) = \mu(\textbf{Supp } A) - \int A\, \mathbf{d}\mu$$

Case 3: Nearly Frank t-norms

Nearly Frank t-norm is a t-norm which can be written as

$$a \underset{\mathbf{N}}{\wedge} b := h^{-1}(h(a) \underset{\mathbf{Fs}}{\wedge} h(b))$$

where $\underset{\mathbf{Fs}}{\wedge}$ is a Frank t-norm and
$h\colon [0,1] \to [0,1]$ is a **negation-preserving automorphism** (**NPA**),
i.e., an increasing bijection satisfying $h(a') = (h(a))'$

a NPA is uniquely determined by its restriction to $[0, 1/2]$

the inverse of a NPA is a NPA

$\underset{\mathbf{Fs}}{\wedge}$ and h are uniquely determined by $\underset{\mathbf{N}}{\wedge}$

nearly Frank $\not\Rightarrow$ Frank (take $h \neq id$)

Case 3: Nearly Frank t-norms II – characterization of measures

Theorem [Barbieri & MN & H. Weber]:
Let $(\mathcal{T}, \underset{\mathbf{N}}{\wedge})$ be a tribe, where $\underset{\mathbf{N}}{\wedge}$ is a strict nearly Frank t-norm. Then all elements of $\mathcal{T}$ are $\mathcal{B}(\mathcal{T})$-measurable and every measure m on $(\mathcal{T}, \underset{\mathbf{N}}{\wedge})$ is of the form

$$\mathbf{m(A) = \mu(\text{Supp}\ A) + \int (h \circ A)\, d\nu}$$

where μ, ν are (classical) measures on $\mathcal{B}(\mathcal{T})$.

$\int (h \circ A)\,\mathbf{d}\nu$ ... **generalized integral measure**

Generated and weakly generated tribes

[Butnariu & Klement; Mesiar; MN]

Problem: For a general strict t-norm $\underset{\cdot}{\wedge}$, we do not know whether all elements of a tribe $(\mathcal{T}, \underset{\cdot}{\wedge})$ are $\mathcal{B}(\mathcal{T})$-measurable.

$\Rightarrow$ **Additional assumption:**
$\mathcal{T}$ is a **weakly generated tribe**, i.e., there is a σ-filter ∇ in $\mathcal{B}(\mathcal{T})$ such that $\mathcal{T}$ is the collection of all functions $A\colon X \to [0,1]$ satisfying

- A is $\mathcal{B}(\mathcal{T})$-measurable,
- $A^{-1}(\{0,1\}) \in \nabla$.

Particular case: $(\nabla = \mathcal{B}(\mathcal{T}))$
$\mathcal{T}$ is a **generated tribe** if $\mathcal{T}$ is the collection of all $\mathcal{B}(\mathcal{T})$-measurable functions $A\colon X \to [0,1]$

Open problem: Is there a strict t-norm $\underset{\cdot}{\wedge}$ and a tribe $(\mathcal{T}, \underset{\cdot}{\wedge})$ which is not weakly generated?

Partial answer: Not for nearly Frank and for many other t-norms.

Case 4: Strict t-norms which are not nearly Frank

Proposition [MN]: **They exist.**

Problem [MN]: **Recognize them!** (By an algorithm.)

Solved by [Mesiar].

Characterization of measures obtained [MN] for **monotonic T-measures on generated tribes**.

Now more generally, for **nonmonotonic T-measures on weakly generated tribes:**

Theorem [Barbieri & MN & H. Weber]:
Let $(\mathcal{T}, \dot{\wedge})$ be a tribe, where $\dot{\wedge}$ is a strict t-norm which is not nearly Frank. Then every measure m on $(\mathcal{T}, \dot{\wedge})$ is a support measure, i.e., it is of the form

$$\boxed{\mathbf{m(A)} = \mu(\textbf{Supp } A)}$$

where μ is a (classical) measure on $\mathcal{B}(\mathcal{T})$.