Regular measures on tribes of fuzzy sets

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Related work presented at Linz Seminars

1979
Henri M. Prade: Nomenclature of fuzzy measures
Erich Peter Klement: Extension of probability measures to fuzzy measures and their characterization
Werner Schwyhla: Conditions for a fuzzy probability measure to be an integral
Josette and Jean-Louis Coulon: Fuzzy boolean algebras

1980
Erich Peter Klement: Some remarks on t-norms, fuzzy $\sigma$-algebras and fuzzy measures
Werner Schwyhla: Remarks on non-additive measures and fuzzy sets
Ulrich Höhle: Fuzzy measures as extensions

1981
Erich Peter Klement: Fuzzy measures assuming their values in the set of fuzzy numbers
1982
Erich Peter Klement: On different approaches to fuzzy probabilities
Didier Dubois: Upper and lower possibilistic expectations and applications
Ronald R. Yager: Probabilities from fuzzy observations

1983
Siegfried Weber: How to measure fuzzy sets

1984
Robert Lowen: Spaces of probability measures revisited

1985
Siegfried Weber: Generalizing the axioms of probability

1986
Erich Peter Klement: Representation of crisp- and fuzzy-valued measures by integrals
Siegfried Weber: Some remarks on the theory of pseudo-additive measures and its applications
1987
Erich Peter Klement: On a class of non-additive measures and integrals

1988
Alain Chateauneuf: Decomposable measures, distorted probabilties and concave capacities
Siegfried Weber: Decomposable measures for conditional objects
Aldo Ventre: A Yosida-Hewitt like theorem for $\bot$-decomposable measures (joint paper with M. Squillante)
Massimo Squillante: $\bot$-decomposable measures and integrals: Convergence and absolute continuity (joint paper with L. D’Apuzzo and R. Sarno)
Ulrich Höhle: Non-classical models of probability theory
1998
Mirko Navara, Pavel Pták: Types of uncertainty and the role of the Frank t-norms in classical and nonclassical logics
Mirko Navara: Nearly Frank t-norms and the characterization of $T$-measures
Giuseppina Barbieri: A representation theorem and a Liapounoff theorem for $T_s$-measures
Beloslav Riečan: On the probability theory and fuzzy sets
Ulrich Höhle: Realizations for generalized probability measures
Marc Roubens: On probabilistic interactions among players in cooperative games
Radko Mesiar: $k$-order pseudo-additive measures
Classical measure theory [Halmos]

THEOREMS about
FUNCTIONALS (MEASURES) on
SETS

Also [Sugeno; Dubois, Prade; Wang, Klir; Pap]
What is fuzzy measure theory?

THEOREMS about

FUZZY FUNCTIONALS (MEASURES) on

SETS

[Feng; Guo, Zhang, Wu]
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[Butnariu, Klement, Mesiar, Barbieri, Weber]
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[Butnariu, Klement, Mesiar, Barbieri, Weber]

Also measure theory on MV-algebras [Cignoli, D’Ottaviano, Mundici, Riečan]
Basic fuzzy logical operations

Standard negation, \( \neg x = 1 - x \)
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**Fuzzy conjunction (t-norm)**: \( T: [0, 1]^2 \rightarrow [0, 1] \) which is commutative, associative, nondecreasing, and \( T(a, 1) = a \)
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A t-norm \( T \) is **strict** iff it is **continuous** and \( x > y, z > 0 \Rightarrow T(x, z) > T(y, z) \)
Basic fuzzy logical operations

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A t-norm $T$ is strict iff it is continuous and

$x > y, z > 0 \Rightarrow T(x, z) > T(y, z)$

Fuzzy disjunction (t-conorm): $S: [0, 1]^2 \rightarrow [0, 1]$ dual to $T$:

$$S(x, y) = \neg T(\neg x, \neg y)$$
### Basic notions of fuzzy measure theory

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<tr>
<th><strong>classical</strong> measure theory</th>
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| \( (A_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}, A_n \not
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| **measure** \( \mu: \mathcal{T} \rightarrow [0, \infty[ \) | **measure** \( \mu: \mathcal{T} \rightarrow [0, \infty[ \) |
| \( \mu(\emptyset) = 0 \) | \( \mu(0) = 0 \) |
| \( \mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) \) | \( \mu(A \dot{\cup} B) = \mu(A) + \mu(B) - \mu(A \cap B) \) * |
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**Always:** Crisp elements of $\mathcal{T}$, i.e., $\mathcal{T} \cap \{0, 1\}^X$, determine a $\sigma$-algebra $\mathcal{B}$

* $(A \cap B)(x) = T(A(x), B(x))$, \hspace{1cm} $(A \cup B)(x) = S(A(x), B(x))$
Full tribes

Example: Let \( \mathcal{B} \) be a \( \sigma \)-algebra of subsets of \( X \), \( \mathcal{T} \) be the corresponding collection of characteristic functions (indicators):

\[
\mathcal{T} = \{ \chi_A | A \in \mathcal{B} \}.
\]

Then \( (\mathcal{T}, T) \) is a tribe (for any t-norm \( T \)).
It is called a **Boolean tribe**.
Full tribes

**Example:** Let $\mathcal{B}$ be a $\sigma$-algebra of subsets of $X$, $\mathcal{T}$ be the corresponding collection of characteristic functions (indicators):

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**Example:** The tribe of all constants from $[0, 1]$ (w.l.o.g., with a singleton domain) may be identified with numbers from $[0, 1]$.

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Example: The tribe of all constants from $[0, 1]$ (w.l.o.g., with a singleton domain) may be identified with numbers from $[0, 1]$. It is called a **full tribe of constants**.

Example: Let $\mathcal{B}$ be a $\sigma$-algebra of subsets of $X$,

$$\mathcal{T} = \{A \in [0, 1]^X \mid A \text{ is } \mathcal{B}\text{-measurable}\}$$

Then $(\mathcal{T}, \mathcal{T})$ is a $T$-tribe for any measurable t-norm $T$. It is called a **full tribe**.
Łukasiewicz t-norm

\[ T_L(x, y) = \max(x + y - 1, 0) \]

These tribes correspond to set-representable \(\sigma\)-complete MV-algebras
Łukasiewicz t-norm

\[ T_L(x, y) = \max(x + y - 1, 0) \]

These tribes correspond to set-representable \( \sigma \)-complete MV-algebras.

**Theorem:** [Butnariu, Klement] All elements of \( T \) are \( B \)-measurable. Each measure is regular and it is of the form

\[ \mu(A) = \int A \, d\nu \]

where \( \nu = \mu \upharpoonright B \) is a (classical) measure on \( B \).
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**Theorem:** [Butnariu, Klement] All elements of \( T \) are \( B \)-measurable. Each measure is **regular** and it is of the form

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\[ \int A \, d\nu \ldots \text{linear integral measure} \]
Frank t-norms

Frank t-norms $T^F_\lambda$, $\lambda \in [0, \infty]$, [Frank] are defined by

$$T^F_\lambda(x, y) = \begin{cases} 
\log_\lambda \left(1 + \frac{\left(\lambda^x - 1\right)\left(\lambda^y - 1\right)}{\lambda - 1}\right) & \text{if } \lambda \in ]0, \infty[ \setminus \{1\}, \\
T_M(x, y) = \min(x, y) & \text{if } \lambda = 0, \\
T_P(x, y) = x \cdot y & \text{if } \lambda = 1, \\
T_L(x, y) = \max(x + y - 1, 0) & \text{if } \lambda = \infty.
\end{cases}$$
Regular measures on full tribes, strict Frank t-norms

Frank t-norm $T^F_{\lambda}$ is strict iff $0 < \lambda < \infty$
Regular measures on full tribes, strict Frank t-norms

Frank t-norm $T^F_\lambda$ is strict iff $0 < \lambda < \infty$

**Theorem:** Regular measures on $(T, T^F_\lambda)$ are (regular) measures on $(T, T_L)$, i.e., of the form

$$\mu(A) = \int A \, d\nu$$

where $\nu = \mu \upharpoonright \mathcal{B}$ is a classical measure on $\mathcal{B}$ ($\mu$ is a linear integral measure).
Nearly Frank t-norms

[Mesiar, MN]

**Nearly Frank t-norm:**

\[ T(a, b) = h_{-1}^{-1}(T_{\chi_T}^F(h_T(a), h_T(b))) \]

where \( T_{\chi_T}^F \) is a Frank t-norm and \( h_T: [0, 1] \to [0, 1] \) is an increasing bijection which *commutes with* \( \neg \), i.e., \( h_T(\neg a) = \neg h_T(a) \)
Nearly Frank t-norms

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\( \lambda_T, h_T \) are uniquely determined by \( T \) (except for the case \( T = T_M \))
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\( \lambda_T, h_T \) are uniquely determined by \( T \) (except for the case \( T = T_M \))

There are nearly Frank t-norms which are not Frank (take \( h_T \neq \text{id} \))
Regular measures on full tribes, strict nearly Frank t-norms

Strict nearly Frank t-norms correspond to strict Frank t-norms
Regular measures on full tribes, strict nearly Frank t-norms

Strict nearly Frank t-norms correspond to strict Frank t-norms

**Theorem:** Each regular measure is of the form

$$\mu(A) = \int (h_T \circ A) \, d\nu$$

where $\nu = \mu \upharpoonright \mathcal{B}$ is a classical measure on $\mathcal{B}$. 
Regular measures on full tribes, strict nearly Frank t-norms

**Strict** nearly Frank t-norms correspond to **strict** Frank t-norms

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\( \int (h_T \circ A) \, d\nu \) ... **generalized integral measure**
Regular measures on full tribes, strict t-norms which are not nearly Frank
Regular measures on full tribes, strict t-norms which are not nearly Frank

There are strict t-norms which are not nearly Frank [MN]
Regular measures on full tribes, strict t-norms which are not nearly Frank

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Can we recognize them?
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Can we recognize them?

Yes \([\text{Mesiar}]\)

**Theorem:** For each strict \( t \)-norm which is not nearly Frank, there are no non-zero regular measures on a full tribe.
Regular measures on full tribes, strict t-norms which are not nearly Frank

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Measures on full tribes, strict t-norms which are not nearly Frank

(Regularity is omitted.)
Measures on full tribes, strict t-norms which are not nearly Frank

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**Theorem:** Each measure is of the form

\[ \mu(A) = \varrho(\text{Supp } A) \]

where \( \varrho \) is a classical measure on \( \mathcal{B} \).
Measures on full tribes, strict t-norms which are not nearly Frank

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**Theorem:** Each measure is of the form

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$\varrho(\text{Supp } A)$ ... support measure
Measures on full tribes,
strict nearly Frank t-norms
Measures on full tribes, strict nearly Frank t-norms

Theorem: [Butnariu, Klement; Mesiar, MN] Each measure is of the form

\[ \mu(A) = \int (h_T \circ A) \, d\nu \pm \varrho(\text{Supp } A) \]

where \( \nu, \varrho \) are classical measures on \( \mathcal{B} \).
Measures on full tribes, strict nearly Frank t-norms

Theorem: [Butnariu, Klement; Mesiar, MN] Each measure is of the form

$$\mu(A) = \int (h_T \circ A) \, d\nu \pm \varrho(\text{Supp } A)$$

where $\nu, \varrho$ are classical measures on $\mathcal{B}$.

Particular case of strict Frank t-norms:

$$\mu(A) = \int A \, d\nu \pm \varrho(\text{Supp } A)$$
Measures on full tribes, strict nearly Frank t-norms

**Theorem:** [Butnariu, Klement; Mesiar, MN] Each measure is of the form

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**Particular case of strict Frank t-norms:**

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Examples of measures on the full tribe of constants \(([0, 1], T)\), \(T\) strict nearly Frank
Examples of measures on the full tribe of constants \(([0, 1], T), T\) strict nearly Frank

Example:

\[
\kappa(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0
\end{cases}
\]

It is a support measure on a singleton.

It is \textbf{monotone}, but \textbf{not regular}.
Examples of measures on the full tribe of constants \(([0, 1], T)\), \(T\) strict nearly Frank

Example:

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\kappa(x) = \begin{cases} 
1 & \text{if } x > 0 \\
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\end{cases}
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It is a support measure on a singleton.
It is **monotone**, but **not regular**.

Example:

\[
\mu(x) = \begin{cases} 
1 - \frac{x}{2} & \text{if } x > 0 \\
0 & \text{if } x = 0 
\end{cases}
\]

It is a measure which is a linear combination of a support measure \(\kappa\) and a regular measure \(\nu = id\), \(\mu = \kappa - \frac{1}{2} id\)
It is **neither monotone nor regular**.
Charges

Alternative notion:

Signed measure (charge) on a tribe \((\mathcal{T}, T)\):

\[ \mu: \mathcal{T} \rightarrow \mathbb{R} \]  

s.t.

- \( \mu(0) = 0 \)
- \( \mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) \)
- \( A_n \uparrow A \Rightarrow \mu(A_n) \rightarrow \mu(A) \)
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Signed measure (charge) on a tribe \((\mathcal{T}, T)\):

\[ \mu: \mathcal{T} \to \mathbb{R} \text{ s.t.} \]

\[ \begin{align*}
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\end{align*} \]

Moreover, for a regular signed measure (regular charge) we require

\[ \begin{align*}
\text{\ding{53}} & \quad A_n \searrow A \Rightarrow \mu(A_n) \to \mu(A)
\end{align*} \]
Tribes which are not full
Tribes which are **not** full

They exist
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Example:
\[ \mathcal{T} = \{ A \in [0, 1]^\mathbb{R} \mid A \text{ is Borel-measurable, } A(x) \in \{0, 1\} \text{ almost everywhere} \} \]
(\( \mathcal{T} \) an arbitrary measurable t-norm)
Tribes which are not full

They exist

**Example:** Boolean tribes

**Example:**
\[ T = \{ A \in [0, 1]^\mathbb{R} \mid A \text{ is Borel-measurable, } A(x) \in \{0, 1\} \text{ almost everywhere} \} \]

\((T\text{ an arbitrary measurable t-norm})\)

**Example:** [Butnariu, Klement; Mesiar; MN]
\[ \mathcal{B} \ldots \text{ a } \sigma\text{-algebra of subsets of } X \]
\[ \Delta \ldots \text{ a } \sigma\text{-ideal in } \mathcal{B} \]
\[ T = \{ A \in [0, 1]^X \mid A \text{ is } \mathcal{B}\text{-measurable, } A^{-1}(]0, 1[) \in \Delta \} \]

\((T, T)\text{ is a tribe called a weakly full tribe (weakly generated tribe)}\)
Tribes which are weakly full

There are many strict t-norms $T$ such that all tribes $(\mathcal{T}, T)$ are weakly full.
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**Example:** Strict t-norms from the following families are sufficient: Aczél–Alsina, Frank, Hamacher*, the eighth Mizumoto, the tenth Mizumoto, Schweizer–Sklar*, the third Schweizer, etc.

* except for one value of the parameter
Tribes which are not weakly full
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**Example:** Hamacher product, $T^H_0$:

$$T^H_0(x, y) = \begin{cases} 
0 & \text{if } x = y = 0, \\
\frac{xy}{x+y-xy} & \text{otherwise}.
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Let $X = \{x, y\}$. There is a distance $d$ on $[0, 1]$ and $c \in \mathbb{R}$ such that $(\mathcal{T}, T^H_0)$ is a tribe, where $\mathcal{T} = \{0, 1\} \cup \{A \in ]0, 1[^X \mid d(A(x), A(y)) \leq c\}$.
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Further counterexamples:

T-norms $T$ obtained from the Hamacher product by the formula

$$T(x, y) = h_{T}^{-1}(T^H_0(h_T(x), h_T(y))) ,$$

where $h_T$ is an order automorphism of $[0, 1]$ which commutes with $\neg$
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- strict Dombi t-norms \((\lambda \in ]0, \infty[)\)

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x \overset{D_{\lambda}}{\wedge} y = \frac{1}{\left(\left(\frac{1}{x} - 1\right)^{\lambda} + \left(\frac{1}{y} - 1\right)^{\lambda}\right)^{\frac{1}{\lambda}}} + 1
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**Theorem:** All other strict t-norms found in the literature are sufficient.
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Open problem: Characterize tribes for non-sufficient t-norms.
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Regular measures on tribes

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Frank (more exactly, nearly Frank) t-norms play a prominent role in the characterization of measures.