Probability and conditional probability on tribes of fuzzy sets

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where ν is a classical probability measure

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- the **classical measure** ν,
- the **domain** of the measure P.

Axiomatic approach 1A: MV-algebras

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= **Łukasiewicz tribes** = systems $\mathcal{T}$ of fuzzy sets s.t.

(T1) $0 \in \mathcal{T}$,

(T2) $A \in \mathcal{T} \Rightarrow A' = 1 - A \in \mathcal{T}$,

(T3) $A, B \in \mathcal{T} \Rightarrow A \odot B \in \mathcal{T}$,

(T4) $(A_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ A_n \nearrow A \Rightarrow A \in \mathcal{T}$,

where $\odot$ is the Łukasiewicz t-norm and $\oplus$ is the Łukasiewicz t-conorm.

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Equivalently, (T3) and (T4) may be replaced by

(T3+) $(A_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}} \Rightarrow \bigodot_n A_n \in \mathcal{T}$.

1A: Measures on MV-algebras

Probability measure (=**state**) $P\colon\mathcal{T}\to[0,1]$ s.t.

(M1) $P(1)=1$,

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Originally [Butnariu, Klement], (T3) and (T4) were replaced by

(T3+) $(A_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}} \Rightarrow \bigodot_n A_n \in \mathcal{T}$,

which is weaker. Nevertheless, it is equivalent in the cases they studied.

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(M2*) is the **valuation property**, stronger than additivity

(M2) $A, B \in \mathcal{T},\ A \odot B = 0 \Rightarrow P(A \oplus B) = P(A) + P(B)$,

which, for t-norms without zero divisors (minimum, product, ...) applies only to fuzzy sets with disjoint supports, $(\mathrm{Supp}\,(A) = A^{-1}(]0,1]))$.

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From now on, we restrict only to **strict** t-norms and their dual t-conorms.

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This happens for the **product** and for **strict Frank** t-norms

$$x \odot y = \log_\lambda \left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right) \text{ for } \lambda \in\,]0, \infty[\setminus \{1\}$$

Nearly Frank t-norms

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Up to the change of scale h, everything remains analogous.

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This can hardly be motivated.

Axiomatic approach 2B: Frank tribes

[Klement, Butnariu, Mesiar, MN, ...]

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All σ-order continuous probability measures are integral measures of the form

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1B and 2B: Comparison

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product as an additional operation	**Łukasiewicz** operations derived
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Only (nearly) Frank t-norms are convenient (as predicted by Butnariu and Klement),
others do not admit σ-order continuous probability measures
(and other measures lack motivation).

Conditional probability (of classical events)

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We obtain the formula for total probability

$$P(A) = P(B)\,P(A|B) + P(B')\,P(A|B')\,.$$

Conditional probability (of classical events)

Probability measure $P(.)$ is considered as a **mixture** of probabilities
$P(.|B)$ (for the case when B occurs),
$P(.|B')$ (for the case when B does not occur).

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$P(.|B), P(.|B')$ are determined by $P(.)$ and $P(B|B') = 0, \quad P(B'|B) = 0,$ having a unique solution for $A = (A \odot B) \oplus (A \odot B')$,

$$P(A) = P(A \odot B) + P(A \odot B')\,,$$

$$P(B)\,P(A|B) = P(A \odot B), \qquad P(B')\,P(A|B') = P(A \odot B')$$

(unless $P(B)$ or $P(B')$ is zero; then the respective conditional probability is not determined).

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This is not only a question of **formulas**, but also **interpretation**.