

Probability and conditional probability on tribes of fuzzy sets

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where ν is a classical probability measure

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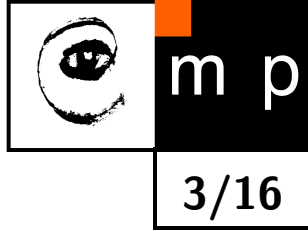
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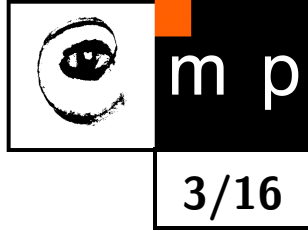
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 - the **classical measure** ν ,
 - the **domain** of the measure P .
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Axiomatic approach 1A: MV-algebras



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Equivalently, (T3) and (T4) may be replaced by

$$(T3+) \ (A_n)_{n \in \mathbb{N}} \in \mathcal{T}^{\mathbb{N}} \Rightarrow \bigodot_n A_n \in \mathcal{T}.$$

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Probability measure (=state) $P: \mathcal{T} \rightarrow [0, 1]$ s.t.

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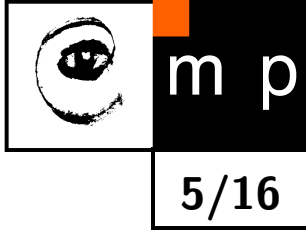
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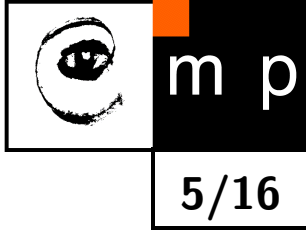
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6/16

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Originally [Butnariu, Klement], (T3) and (T4) were replaced by

$$(T3+) \ (A_n)_{n \in \mathbb{N}} \in \mathcal{T}^{\mathbb{N}} \Rightarrow \odot_n A_n \in \mathcal{T},$$

which is weaker. Nevertheless, it is equivalent in the cases they studied.

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(M2*) is the **valuation property**, stronger than additivity

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which, for t-norms without zero divisors (minimum, product, ...) applies only to fuzzy sets with disjoint supports, ($\text{Supp}(A) = A^{-1}(]0, 1])$).

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From now on, we restrict only to **strict** t-norms and their dual t-conorms.

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This happens for the **product** and for **strict Frank** t-norms

$$x \odot y = \log_{\lambda} \left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1} \right) \text{ for } \lambda \in]0, \infty[\setminus \{1\}$$

Nearly Frank t-norms

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Up to the change of scale h , everything remains analogous.

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This can hardly be motivated.

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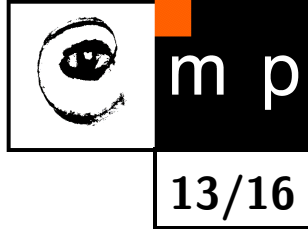
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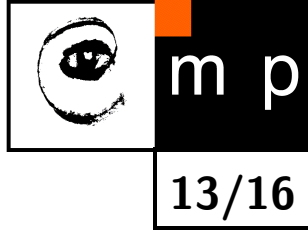
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(M4) does not admit support measures.

2B: Characterization of σ -order continuous measures on Frank tribes

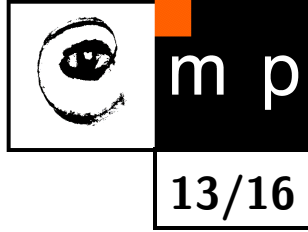


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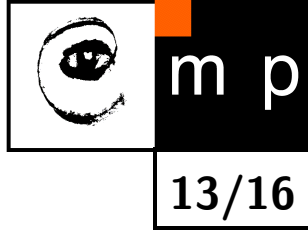
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All elements of \mathcal{T} are $\mathcal{B}(\mathcal{T})$ -measurable.

2B: Characterization of σ -order continuous measures on Frank tribes

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All σ -order continuous probability measures are integral measures of the form

$$P(A) = \int A d\nu,$$

where $\nu = P \upharpoonright \mathcal{B}(\mathcal{T})$ is a classical probability measure.

1B and 2B: Comparison

1B	2B
Łukasiewicz tribe	product or strict Frank tribe
product as an additional operation	Łukasiewicz operations derived
(M2) $A \odot B = 0 \Rightarrow$ $P(A \oplus B) = P(A) + P(B)$	(M2*) $P(A \oplus B)$ $= P(A) + P(B) - P(A \odot B)$
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Only (nearly) Frank t-norms are convenient (as predicted by Butnariu and Klement), others do not admit σ -order continuous probability measures (and other measures lack motivation).

Conditional probability (of **classical** events)

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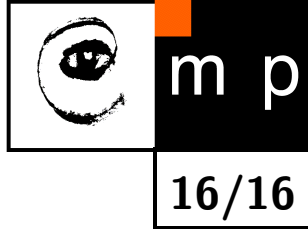
$P(\cdot|B), P(\cdot|B')$ are determined by $P(\cdot)$ and $P(B|B') = 0, P(B'|B) = 0$, having a unique solution for $A = (A \odot B) \oplus (A \odot B')$,

$$P(A) = P(A \odot B) + P(A \odot B'),$$

$$P(B) P(A|B) = P(A \odot B), \quad P(B') P(A|B') = P(A \odot B')$$

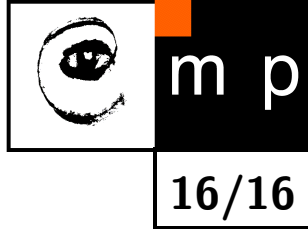
(unless $P(B)$ or $P(B')$ is zero; then the respective conditional probability is not determined).

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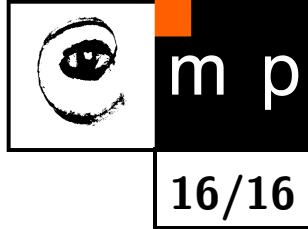
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This is not only a question of **formulas**, but also **interpretation**.