



When Do We Need Quantum Probability Instead of the Classical One

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Jak nevyšly volební předpovědi v ČR 2013



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http://zpravy.idnes.cz/jak-se-trefily-pruzkumy-0wa-/domaci.aspx?c=A131027_120534_domaci_jw

Rozsah	Volby	Factum	Median	STEM	CVVM	Sanep	Médea	Aisa
		1000	1000	1000	1000	7000	1000	1000
ČSSD	20.5	22.8	25.5	25.9	26	23.8	22.2	23
ANO	18.7	12.1	13	16.1	16.5	11.6	16.9	16
KSČM	14.9	17.1	16	13.3	18	16.9	11.8	14
TOP 09	12	13.2	13	11.5	9	11.9	9.6	10.5
ODS	7.7	7.2	8	8.6	6.5	7.5	5.5	7
Úsvit	6.9	3.7	4	5.9	5	5.3	8.2	6
KDU	6.8	5.9	6	4.5	5	5.7	6.2	6
SZ	3.2	4.3	3	2.6	2	3.1	2.9	3
SPOZ	1.5	4.7	5	2.6	3.5	5.2	3.7	4
ostatní	7.8	9	6.5	9	8.5	9	13	10.5
Účast	59.5	54	60	67	63	59.3	71	78
χ^2 -test		≈ 0	≈ 0	$5 \cdot 10^{-6}$	$2 \cdot 10^{-12}$	≈ 0	≈ 0	$3 \cdot 10^{-10}$

Tučně jsou vyznačeny předpovědi, které se vejdu do 95% symetrického intervalového odhadu. χ^2 -**test dobré shody** ukazuje, že u nejúspěšnější agentury lze hypotézu o shodě rozdělení zamítnout na hladině významnosti $5 \cdot 10^{-6}$, tj. takový výsledek bychom při shodě rozdělení dostali s pravděpodobností 1/200 000, u ostatních agentur ještě menší.

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σ -algebra of sets, $\mathcal{L} \subseteq 2^U$:

- $U \in \mathcal{L}$
- $A \in \mathcal{L} \implies U \setminus A \in \mathcal{L}$
- $\{A_i \mid i \in \mathbb{N}\} \subseteq \mathcal{L} \implies \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{L}$

State (=probability measure) $s: \mathcal{L} \rightarrow [0, 1]$:

- $s(U) = 1$
- $\{A_i \mid i \in \mathbb{I}\} \subseteq \mathcal{L}, A_i \cap A_j = \emptyset \text{ for } i \neq j \implies s\left(\bigcup_{i \in \mathbb{I}} A_i\right) = \sum_{i \in \mathbb{I}} s(A_i)$

finitely, resp. **countably** additive for $\mathbb{I}$ finite, resp. countable

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State space (of **finitely** additive states): $\mathcal{S}(\mathcal{L}) \subseteq [0, 1]^{\mathcal{L}}$ compact, convex, Choquet simplex

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Pure states: extreme points of $\mathcal{S}(\mathcal{L})$

Two-valued states: $\mathcal{S}(\mathcal{L}) \cap \{0, 1\}^{\mathcal{L}} =$ points in the Stone space

Here: two-valued = pure

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Two-valued states: $\mathcal{S}(\mathcal{L}) \cap \{0, 1\}^{\mathcal{L}} =$ points in the Stone space

Here: two-valued = pure

We need **disjoint** unions, not **all** unions! $\implies$ **σ -class** of sets, $\mathcal{L} \subseteq 2^U$:

- $U \in \mathcal{L}$
- $A \in \mathcal{L} \implies U \setminus A \in \mathcal{L}$
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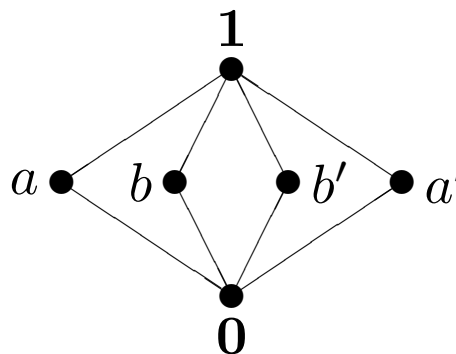
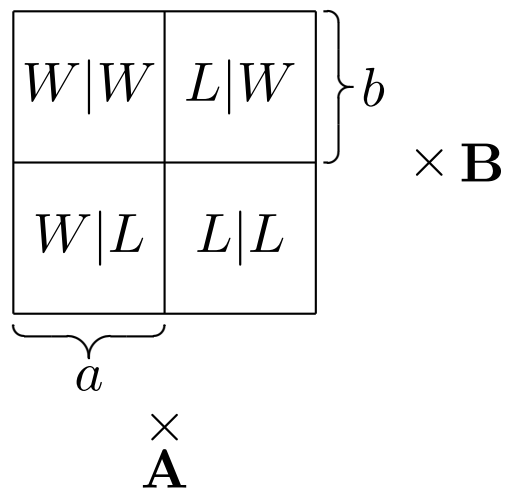
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Example 1 (transparent barriers)



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W/L = Wins/Loses with|without player JJ



$$U = \{W|W, W|L, L|W, L|L\}$$

$$A = \{\emptyset, \underbrace{\{W|W, W|L\}}_a, \underbrace{\{L|W, L|L\}}_{a'}, U\}$$

$$B = \{\emptyset, \underbrace{\{W|W, L|W\}}_b, \underbrace{\{W|L, L|L\}}_{b'}, U\}$$

$$\mathcal{L} = A \cup B = \{\emptyset, a, a', b, b', U\}$$

$\mathcal{L}$ is a (nondistributive modular) lattice called *MO2*

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Example 2 (transparent barriers)



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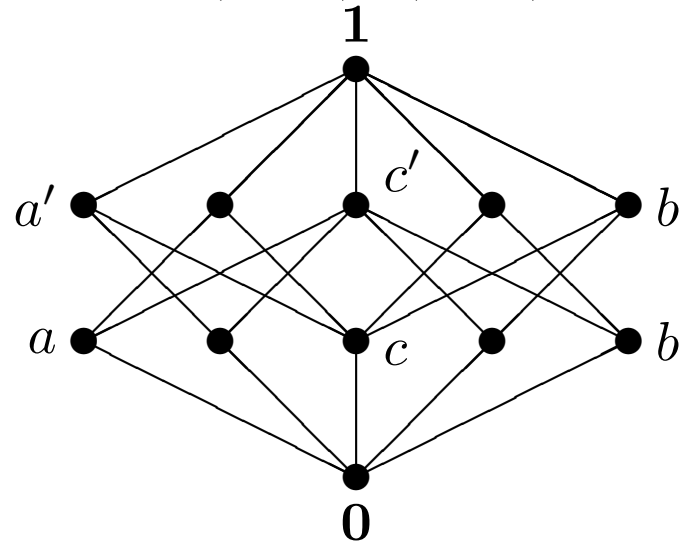
Example 1 with one more result, $c = \text{match cancelled}$

$$A = \{0, a, c, (a \vee c)', a \vee c, a', c', 1\}$$

$$B = \{0, b, c, (b \vee c)', b \vee c, b', c', 1\}$$

$$A \cap B = \{0, c, c', 1\}$$

$$\mathcal{L} = A \cup B = \{0, a, b, c, a \vee c, b \vee c, (a \vee c)', (b \vee c)', a', b', c', 1\}$$



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Example 3 (non-transparent barriers)



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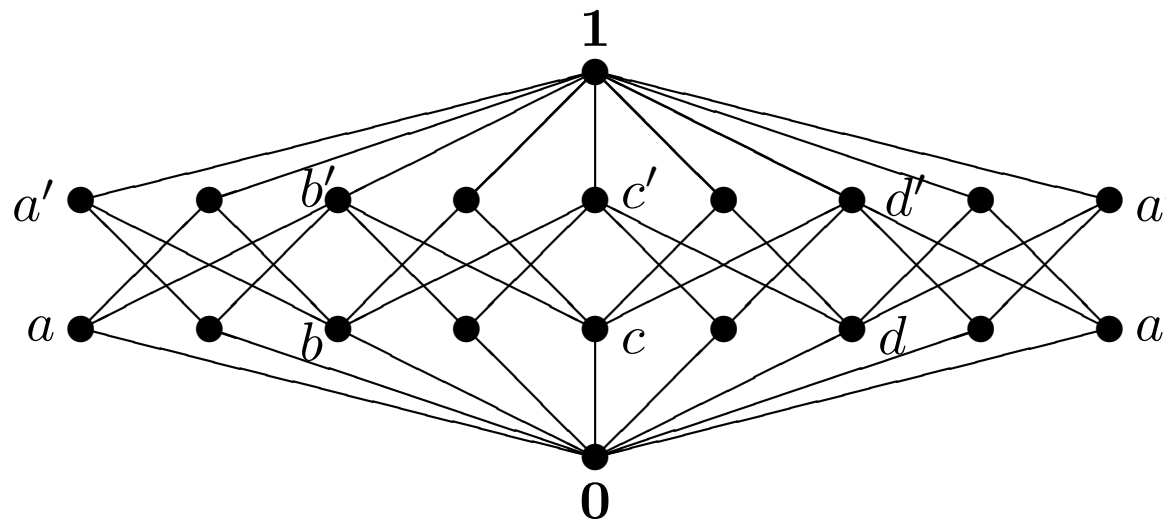
$\mathbf{D}$
 $\times$

$\mathbf{C} \times$

d	a
c	b

$\times \mathbf{A}$

$\times$
 $\mathbf{B}$





Definition: An **orthomodular poset** is a poset with bounds $0, 1$ equipped with a unary operation $' : \mathcal{L} \rightarrow \mathcal{L}$ (**orthocomplementation**) such that, for all $a, b \in \mathcal{L}$,

- $a'' = a$
- $a \leq b \implies b' \leq a'$
- $a \wedge a' = 0$
- $a \leq b \implies b = a \vee (a' \wedge b)$ (**orthomodular law**, existence of rhs assumed)

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Orthogonality: $a \perp b \iff a \leq b'$

$\implies a \wedge b = 0$ (This condition is strictly stronger than the usual $a \wedge b = 0$.)

Usually we assume σ -completeness, i.e., each sequence of mutually orthogonal elements has a supremum.

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Example: A σ -class of subsets is an orthomodular poset. If it is a lattice, it is an orthomodular lattice.

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Example: A σ -class of subsets is an orthomodular poset. If it is a lattice, it is an orthomodular lattice.

State: $s: \mathcal{L} \rightarrow [0, 1]$ such that

- $s(\mathbf{1}) = 1$
- $\{a_i \mid i \in \mathbb{I}\} \subseteq \mathcal{L}, a_i \perp a_j \text{ for } i \neq j \implies s\left(\bigvee_{i \in \mathbb{I}} a_i\right) = \sum_{i \in \mathbb{I}} s(a_i)$

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Boolean subalgebra: $\mathcal{M} \subseteq \mathcal{L}$ such that

- $0, 1 \in \mathcal{M}$,
- $a \in \mathcal{M} \implies a' \in \mathcal{M}$,
- $(\mathcal{M}, \leq|_{\mathcal{M}}, ' |_{\mathcal{M}})$ is a Boolean algebra.

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Theorem: Every OMP is the union of its maximal Boolean subalgebras.

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Compatibility: $a \leftrightarrow b \iff \exists$ Boolean subalgebra $\mathcal{M}$: $a, b \in \mathcal{M}$

Atom: $a \in \mathcal{L} \setminus \{0\}$ such that there is no b satisfying $0 < b < a$

$\mathcal{A}(\mathcal{L}) :=$ the set of all atoms of $\mathcal{L}$

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Proposition: The space of finitely additive states is convex and compact (in the product topology).

Theorem: [Shultz 74, MN 92] **Every** compact convex subset of a locally convex topological linear space is affinely homeomorphic to the space of finitely additive (as well as all) states of some OML.

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In particular, the state space may be empty.

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In particular, the state space may be empty.

This is a mathematical peculiarity, for physical applications we need “enough” states:

Definition: An OMP L is **rich** iff $\forall a, b \in L, a \not\leq b \exists \text{state } s : s(a) = 1 = s(b)$.

Theorem: All rich OMLs form a variety.

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$$s(a) + s(b) - s(a \wedge b) \leq 1$$

$$0 \geq s(a \wedge b) + s(b \wedge c) + s(c \wedge d) - s(a \wedge d) - s(b) - s(c)$$

$$s(a) + s(b) + s(c) - s(a \wedge b) - s(a \wedge c) - s(b \wedge c) \leq 1$$

$$s(a \wedge b) + s(b \wedge c) + s(c \wedge d) - s(a \wedge d) - s(b) - s(c) \geq -1$$

The first is equivalent to the **valuation property**:

$$s(a \wedge b) + s(a \vee b) = s(a) + s(b)$$

If a rich OML is not a Boolean algebra, all Bell inequalities are violated.

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H ... a separable Hilbert space (real or complex)

$L(H)$... the set of all closed subspaces of H (equivalently, all projectors of H)

$$\mathbf{0} = \{0\},$$

$$\mathbf{1} = H,$$

$$A \leq B \iff A \subseteq B,$$

$$A \wedge B = A \cap B,$$

$$A' = \{x \in H \mid \forall y \in A : y \perp x\},$$

$$A \vee B = \text{Lin}(A \cup B),$$

where Lin denotes the closed linear hull

For projectors, compatibility is equivalent to commutation, $PQ = QP$, therefore the term “*noncommutative measure theory*”

Problem [Birkhoff, von Neumann, Mackey]: Find (well-motivated, verifiable) properties which characterize Hilbert lattices among OMLs.

Partial results: [Pulmannová], [Bunce, J.D.M. Wright] ...

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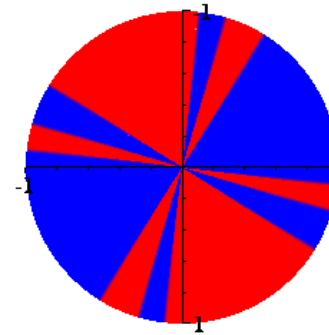
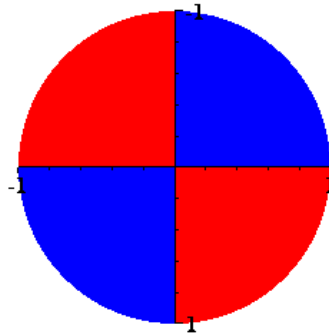
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The only restriction of states for $\dim P = 1$: $s(P') = 1 - s(P)$

Many two-valued states = colourings of non-zero vectors by two colors (blue, red) such that each orthogonal basis contains exactly one red vector



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1. For $q \in H$, $\|q\| = 1$, define a **vector state**

$$s_q(\text{Lin}(\{y_1, \dots, y_n\})) = \sum_{i=1}^n (q \cdot y_i)^2 = \sum_{i=1}^n \cos^2 \angle(q, y_i)$$

for any orthonormal basis $(y_1, \dots, y_n)$ of H

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Corollary: $s_q(q) = 1$

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Corollary: $s_q(q) = 1$

2. **Mixture** of vector states

$$s(P) = \sum_i c_i s_{q_i}(P), \text{ where } c_i > 0, \sum_i c_i = 1 .$$

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3. What else?

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Corollary: $s_q(q) = 1$

2. **Mixture** of vector states $s(P) = \sum_i c_i s_{q_i}(P)$, where $c_i > 0$, $\sum_i c_i = 1$.

3. What else?

Nothing!

Gleason's Theorem [Gleason 57]: For $\dim H \geq 3$, all states are mixtures of vector states.

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Crucial case: $H = \mathbb{R}^3$ (simplified proof by [Cooke, Keane, Moran 85]).

Corollary 1: The restriction of a state to 1D subspaces is continuous (proved by [von Neumann 1932] even for $\mathbb{R}^2$, error found by [Hermann 1935],

Corollary 2: A finitely-valued state is constant on 1D subspaces, i.e., it is a dimension function.

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Crucial corollary: There is no two-valued state (=hidden variable) (answer to a question by [Einstein, Podolsky, Rosen 35], simplified proofs by [Bell 64, Bell 66], [Kochen, Specker 1967], ... , [Peres 1996]).

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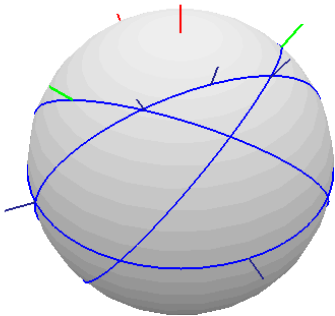
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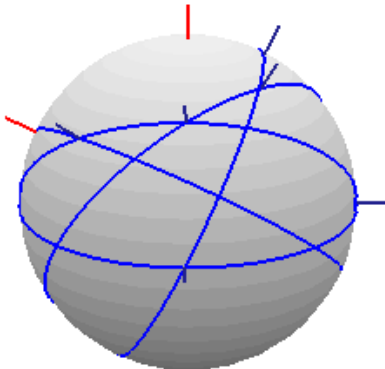
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BGL



MGL

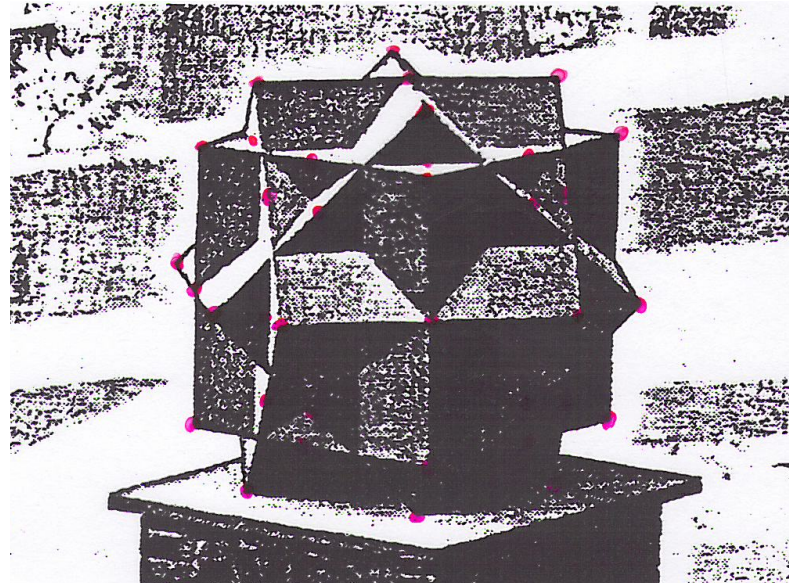


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It is possible to find a finite set of vectors whose orthogonality relations exclude the possibility of a two-valued state.

The smallest example known uses 31 vectors, the following uses 33 vectors:



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Theorem: [Cabello] There is no two-valued state on $\mathcal{L}(\mathbb{R}^4)$.

Take 36 vectors in $\mathbb{R}^4$ ($\bar{1}$ denotes -1):

1000	1000	0100	1111	1111	111 $\bar{1}$	11 $\bar{1}\bar{1}$	111 $\bar{1}$	11 $\bar{1}1$
0100	0010	0010	11 $\bar{1}\bar{1}$	1 $\bar{1}1\bar{1}$	11 $\bar{1}1$	1 $\bar{1}1\bar{1}$	1 $\bar{1}11$	1 $\bar{1}11$
0011	0101	1001	1 $\bar{1}00$	10 $\bar{1}0$	1 $\bar{1}00$	1001	10 $\bar{1}0$	100 $\bar{1}$
001 $\bar{1}$	010 $\bar{1}$	100 $\bar{1}$	001 $\bar{1}$	010 $\bar{1}$	0011	0110	0101	0110

Each of the 9 column represents an orthogonal basis of $\mathbb{R}^4$ and each vector **occurs twice**. The number of vectors of unit state in this table must be both even and odd (9)—a contradiction.

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Theorem: ([Pták, MN 04], extended by [Harding, Jager, Smith])
If $n \geq 4$, then there is no non-trivial $\mathbb{Z}_2$ -valued **state** on $\mathcal{L}(\mathbb{R}^n)$.
Another proof in [Peres 1996].

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Problem [Peres 1996]: Are there non-trivial $\mathbb{Z}_2$ -valued **states** on $\mathcal{L}(\mathbb{R}^3)$?

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$\iff$ is there a colouring of non-zero vectors in $\mathbb{R}^3$ by two colors (blue, red) such that both colours are used and each orthogonal basis contains an odd number of red vectors?

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Answer: NO. [Voráček & MN, 2021].

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Answer: NO. [Voráček & MN, 2021].

Notation of an orthogonal basis:



Each such edge must contain an odd number of “red” vertices (vectors evaluated by 1).

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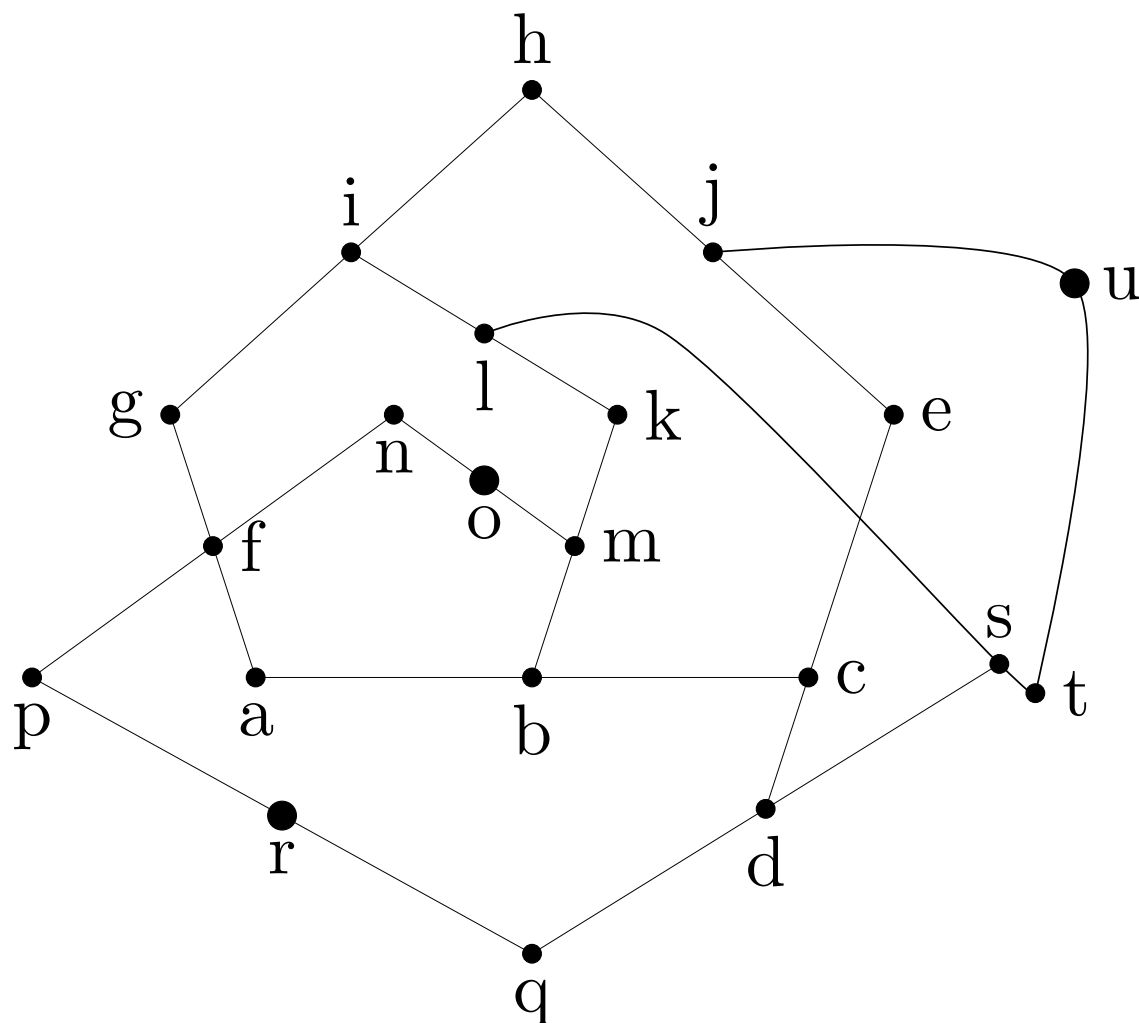
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There are vectors in $\mathbb{R}^3$ satisfying these orthogonality relations (Greechie diagram).

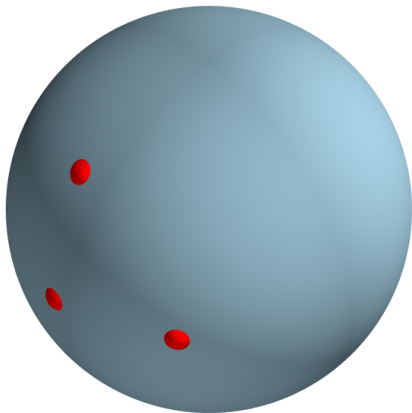
There are 13 edges.

Except for $\mathbf{r}, \mathbf{u}, \mathbf{o}$, each vector is contained in 2 edges.

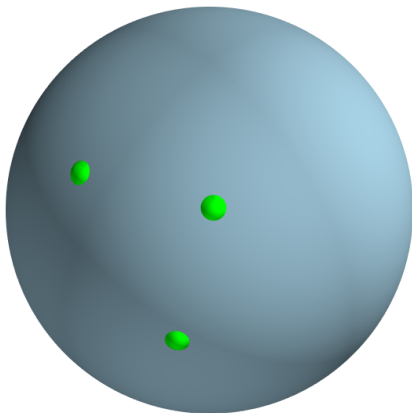
Vectors $\mathbf{r}$, $\mathbf{u}$, $\mathbf{o}$ are not orthogonal, but satisfy the same equation:

$$\begin{aligned} & \overbrace{s(\mathbf{a}) \oplus s(\mathbf{b}) \oplus s(\mathbf{c})}^{=1} \oplus \overbrace{s(\mathbf{a}) \oplus s(\mathbf{f}) \oplus s(\mathbf{g})}^{=1} \oplus \dots = 13 = 1 \pmod{2} \\ &= s(\mathbf{r}) \oplus s(\mathbf{u}) \oplus s(\mathbf{o}) \end{aligned}$$

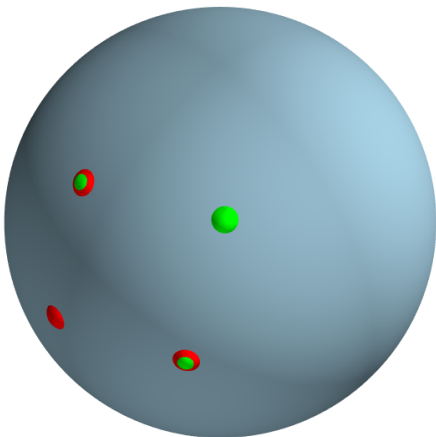
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$$s(\mathbf{r}) \oplus s(\mathbf{u}) \oplus s(\mathbf{o}) = 1$$



$$s(\mathbf{r}) \oplus s(\mathbf{u}) \oplus s(\mathbf{o}') = 1$$



$$\overbrace{s(\mathbf{r}) \oplus s(\mathbf{u}) \oplus s(\mathbf{o})}^1 \oplus \overbrace{s(\mathbf{r}) \oplus s(\mathbf{u}) \oplus s(\mathbf{o}')}^1 = 0 = s(\mathbf{o}) \oplus s(\mathbf{o}') \\ s(\mathbf{o}) = s(\mathbf{o}')$$

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A valid deduction in the classical logic: 16 formulas in 11 variables

$$d_1 \rightarrow \neg b_2$$

$$d_1 \rightarrow \neg b_3$$

$$d_2 \rightarrow a_2 \vee b_2$$

$$d_2 \rightarrow \neg b_3$$

$$d_3 \rightarrow \neg b_2$$

$$d_3 \rightarrow (a_1 \vee a_2 \rightarrow b_3)$$

$$d_4 \rightarrow a_2 \vee b_2$$

$$d_4 \rightarrow (a_1 \vee a_2 \rightarrow b_3)$$

$$(a_2 \vee c_1) \vee (b_3 \vee d_1)$$

$$(a_2 \vee c_2) \vee (a_1 \vee b_1 \rightarrow d_1)$$

$$c_2 \rightarrow b_3 \vee d_2$$

$$c_1 \rightarrow b_1 \vee d_2$$

$$(a_2 \vee c_1) \vee ((a_1 \vee a_2 \rightarrow b_3) \rightarrow d_3)$$

$$(a_2 \vee c_2) \vee (b_1 \vee d_3)$$

$$c_2 \rightarrow ((a_1 \vee a_2 \rightarrow b_3) \rightarrow d_4)$$

$$c_1 \rightarrow (a_1 \vee b_1 \rightarrow d_4)$$

imply

$$(a_1 \rightarrow a_2) \vee b_1$$

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The above deduction is invalid in quantum logic:
In the Hilbert lattice $\mathcal{L}(\mathbb{R}^3)$, we take

- $a_1 = (1, 0, 0)$
- $a_2 = (0, 1, 0)$
- $b_1 = (0, 1, 1)$
- $b_2 = (1, 0, 1)$
- $b_3 = (1, 1, 0)$
- $c_1 = (1, 0, 2)$
- $c_2 = (2, 0, 1)$
- $d_1 = (-1, 1, 1)$
- $d_2 = (1, -1, 1)$
- $d_3 = (1, 1, -1)$
- $d_4 = (1, 1, 1)$

Then $(a_1 \rightarrow a_2) \vee b_1 = (1, 0, 0)'$

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We need only 3 variables and 9 formulas

$$(a \vee c)' \vee b' \wedge (b \vee c) \vee (a \vee b)' \vee c' \wedge (a \vee b)$$

$$a' \wedge (a \vee c) \vee (b \vee c)' \vee (a \vee b)' \vee c' \wedge (a \vee b)$$

$$a' \wedge (a \vee c) \vee b' \wedge (b \vee c) \vee (a \vee b)' \vee c$$

$$b \vee b' \wedge (c \vee (a \vee c)') \vee (a \vee b) \wedge (c \vee (a \vee b)') \vee (a \vee c) \wedge (b \vee (b \vee c)')$$

$$b \vee b' \wedge (c \vee c' \wedge (a \vee b)) \vee (a \vee c)' \vee (a \vee c) \wedge (b \vee (b \vee c)')$$

$$b \vee c' \wedge (c \vee (a \vee b)') \vee (a \vee b) \wedge (c \vee (a \vee b)') \vee (b \vee c) \wedge (a \vee (a \vee c)')$$

$$b \vee c' \wedge (a \vee c) \vee a' \wedge (a \vee c) \vee (b \vee c) \wedge (a \vee (a \vee c)')$$

$$b \vee b' \wedge (c \vee (a \vee c)') \vee c' \wedge (a \vee b) \vee (a \vee (a \vee c)') \wedge (b \vee (b \vee c)')$$

$$b \vee b' \wedge (c \vee c' \wedge (a \vee b)) \vee a' \wedge (a \vee c) \vee (a \vee (a \vee c)') \wedge (b \vee (b \vee c)')$$

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$$b \vee b' \wedge (c \vee (a \vee c)') \vee c' \wedge (a \vee b) \vee (a \vee (a \vee c)') \wedge (b \vee (b \vee c)')$$

$$b \vee b' \wedge (c \vee c' \wedge (a \vee b)) \vee a' \wedge (a \vee c) \vee (a \vee (a \vee c)') \wedge (b \vee (b \vee c)')$$

These imply a' only in the classical logic (the same counterexample in a Hilbert lattice works).

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