Introduction to Fuzzy Logic

Mirko Navara
Center for Machine Perception
Faculty of Electrical Engineering
Czech Technical University in Prague
Czech Republic
http://cmp.felk.cvut.cz/˜navara/

January 9, 2024

CLASSICAL LOGIC (CL)

SYNTAX OF CLASSICAL LOGIC 1

$\mathcal{A}$ ... countable set of propositional variables
$\mathcal{L} = \{\to, \mathbf{0}\}$ ... the set of logical connectives:
$\to$ ... (binary) implication
$\mathbf{0}$ ... (nulary) false
Formulas

- all elements of $\mathcal{A}$ are formulas
- $\mathbf{0}$ is a formula
- if A, B are formulas, then $A \to B$ is a formula

More exactly, we use brackets like $(A) \to (B)$
Derived connectives:
$\neg A = A \to \mathbf{0}$ ... (unary) negation
$\mathbf{1} = \neg\mathbf{0} = \mathbf{0} \to \mathbf{0}$ ... (nulary) true
$A \wedge B = \neg(A \to \neg B)$ ... (binary) conjunction
$A \vee B = \neg A \to B$ ... (binary) disjunction
$A \leftrightarrow B = (A \to B) \wedge (B \to A)$ ... (binary) equivalence

SEMANTICS OF CLASSICAL LOGIC

In general: a Boolean algebra, it is enough to consider
Standard semantics

$\{0, 1\}$ ... the set of truth values
$\to$ ... (interpreted as) Boolean implication $\Rightarrow$
$\mathbf{0}$ ... (interpreted as) 0
Interpretation of derived connectives:
$\neg$ ... Boolean negation
$\mathbf{1}$ ... 1
$\wedge$ ... conjunction
$\vee$ ... disjunction
$\leftrightarrow$ ... Boolean equivalence $\Leftrightarrow$
An **evaluation** (**truth assignment**) can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Tautology is a formula A which is *always* evaluated to 1
Notation: $\models A$
Moreover, for any set of formulas $\mathcal{T}$,
$\mathcal{T} \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in \mathcal{T} : e(B) = 1$.
Contradiction is a formula which is *always* evaluated to 0
A formula is **satisfiable** if it is evaluated to 1 for *at least one evaluation*

SYNTAX OF CLASSICAL LOGIC 2

Logical axioms

$$
\begin{array}{ll}
(\mathrm{C1}) & A \to (B \to A) \\
(\mathrm{C2}) & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\
(\mathrm{C3}) & (\neg A \to \neg B) \to (B \to A)
\end{array}
$$

Deduction rule: Modus Ponens

$$\mathrm{MP}(A,\ A \to B): \quad \frac{A,\ A \to B}{B}$$

Theory $\mathcal{T}$ ... set of formulas (**special axioms**)
Provable formula (=**theorem**) in theory $\mathcal{T}$ is a formula which admits a **proof**, i.e., a finite sequence of formulas such that each of them is

- a special axiom (=element of $\mathcal{T}$), or
- an instance of a logical axiom (obtained by a substitution), or
- a result of application of a deduction rule to preceding formulas in the proof.

Notation:
$\mathcal{T} \vdash A$
$B \vdash A$ (for $\mathcal{T} = \{B\}$)
$\vdash A$ (for $\mathcal{T} = \emptyset$)

Example Cl1

$A \vdash B \to A$

$$
\begin{array}{rl}
(\mathrm{C1}): & D_1 = A \to (B \to A) \\
\text{SA (special axiom)}: & D_2 = A \\
\mathrm{MP}(D_2, D_1): & D_3 = B \to A
\end{array}
$$

$\Rightarrow$ we can add a deduction rule RI(A): $\dfrac{A}{B \to A}$

Example Cl2

$\vdash A \to A$
For brevity, let B denote any provable formula, e.g., axiom (C1).

$$
\begin{array}{rl}
(\mathrm{C1}): & D_1 = B \\
\mathrm{RI}(D_1): & D_2 = A \to B \\
(\mathrm{C2}), C := A: & D_3 = (A \to (B \to A)) \to ((A \to B) \to (A \to A)) \\
(\mathrm{C1}): & D_4 = A \to (B \to A) \\
\mathrm{MP}(D_4, D_3): & D_5 = (A \to B) \to (A \to A) \\
\mathrm{MP}(D_2, D_5): & D_6 = A \to A
\end{array}
$$

$\Rightarrow$ we can add an axiom (AA): $A \to A$

Corollary Cor1

$\vdash \mathbf{0} \to \mathbf{0}, \qquad \vdash \neg\mathbf{0}, \qquad \vdash \mathbf{1}$

Example Cl3

$\vdash A \to \mathbf{1}$ for all A

$$\begin{aligned} \text{Cor1}: &\quad D_1 = \mathbf{1} \\ \text{RI}(D_1): &\quad D_2 = A \to \mathbf{1} \end{aligned}$$

Example Cl4

$\{B, \neg B\} \vdash A$ for all A

$$\begin{aligned} \text{(C3)}: &\quad D_1 = (\neg A \to \neg B) \to (B \to A) \\ \text{SA}: &\quad D_2 = \neg B \\ \text{RI}(D_2): &\quad D_3 = \neg A \to \neg B \\ \text{MP}(D_3, D_1): &\quad D_4 = B \to A \\ \text{SA}: &\quad D_5 = B \\ \text{MP}(D_5, D_4): &\quad D_6 = A \end{aligned}$$

$\Rightarrow$ we can add a deduction rule ALL(B) : $\dfrac{B, \neg B}{A}$

Example Cl5

$\{A \to B, B \to C\} \vdash A \to C$

$$\begin{aligned} \text{(SA1)}: &\quad D_1 = A \to B \\ \text{(SA2)}: &\quad D_2 = B \to C \\ \text{RI}(D2): &\quad D_3 = A \to (B \to C) \\ \text{(C2)}: &\quad D_4 = (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\ \text{MP}(D_3, D_4): &\quad D_5 = (A \to B) \to (A \to C) \\ \text{MP}(D_1, D_5): &\quad D_6 = A \to C \end{aligned}$$

$\Rightarrow$ we can add a deduction rule TI$(A \to B, B \to C)$: $\dfrac{A \to B, B \to C}{A \to C}$
(transitivity of implication)

Example Cl6

$\{E \to T, E \to \neg T\} \vdash \neg E$

$$\begin{aligned} \text{(SA1)}: &\quad D_1 = E \to T \\ \text{(SA2)}: &\quad D_2 = E \to \neg T = E \to (T \to \mathbf{0}) \\ \text{(C2)}: &\quad D_3 = (E \to (T \to \mathbf{0})) \to ((E \to T) \to (E \to \mathbf{0})) \\ \text{MP}(D_2, D_3): &\quad D_4 = (E \to T) \to (E \to \mathbf{0}) \\ \text{MP}(D_1, D_4): &\quad D_5 = E \to \mathbf{0} = \neg E \end{aligned}$$

Example Cl7

$\vdash \mathbf{0} \to A$ for all A (*ex falso quodlibet*)

$$\begin{aligned} \text{(C3) } B := \mathbf{0}: &\quad D_1 = (\neg A \to \neg\mathbf{0}) \to (\mathbf{0} \to A) \\ \text{Cl3, } A := \neg A: &\quad D_2 = \neg A \to \neg\mathbf{0} \\ \text{MP}(D_2, D_1): &\quad D_3 = \mathbf{0} \to A \end{aligned}$$

$\Rightarrow$ we can add an axiom $\mathbf{0} \to A$

Example Cl8

$\vdash A \vee \neg A$ for all A (*tertium non datur*)

$$\text{Cl2, } A := \neg A: \qquad D_1 = \neg A \to \neg A = A \vee \neg A$$

$\Rightarrow$ we can add an axiom $A \vee \neg A$ (in contrast to $\neg A \vee A$ which is also provable, but not yet proved)

Example Cl9

$B \vdash A \vee B$ for all A, B

$$\text{RI}(B): \qquad D_1 = \neg A \to B = A \vee B$$

$\Rightarrow$ we can add a deduction rule $\dfrac{B}{A \vee B}$ (in contrast to $\dfrac{A}{A \vee B}$ which is also correct, but not yet proved)

Deduction theorem in classical logic

$\mathcal{T}$ ... theory
A, B ... formulas
$\mathcal{T} \cup \{A\} \vdash B$ iff $\mathcal{T} \vdash A \to B$

Proof

$\Leftarrow$:

$$\begin{array}{rl} & \text{//BEGIN of proof of } \mathcal{T} \vdash A \to B \\ & \vdots \\ & D_{i-1} = A \to B \\ & \text{//END of proof of } \mathcal{T} \vdash A \to B \\ \text{SA}: & D_i = A \\ \text{MP}(D_i, D_{i-1}): & D_{i+1} = B \end{array}$$

$\Rightarrow$: Proof by contradiction: Suppose that there is a formula B such that $\mathcal{T} \cup \{A\} \vdash B$, $\mathcal{T} \nvdash A \to B$.

1. B is neither an axiom, nor a special axiom ($\in \mathcal{T}$) because then $\mathcal{T} \vdash B$,

$$\begin{array}{rl} & D_1 = B \\ RI(D_1): & D_2 = A \to B \end{array}$$

hence $\mathcal{T} \vdash A \to B$.

2. $B \neq A$ because $\mathcal{T} \vdash A \to A$.

3. B is obtained by deduction in the proof of $\mathcal{T} \cup \{A\} \vdash B$.

WLOG, we choose for B a formula with the shortest possible proof; its shortest proof must be of the following form:

$$\begin{array}{rl} & \vdots \\ & D_i \\ & \vdots \\ & D_j = D_i \to B \\ & \vdots \\ \text{MP}(D_i, D_j): & D_m = B \end{array}$$

for $i < j < m$ or $j < i < m$.

The proofs of $\mathcal{T} \cup \{A\} \vdash D_i$, $\mathcal{T} \cup \{A\} \vdash D_j$ are of lengths $< m$, therefore

$$\mathcal{T} \vdash A \to D_i$$
$$\mathcal{T} \vdash A \to D_j = A \to (D_i \to B)$$

Proof of $\mathcal{T} \vdash A \to B$:

$$\begin{aligned}
& //\text{BEGIN of proof of } \mathcal{T} \vdash A \to D_i \\
& \vdots \\
& D_k = A \to D_i \\
& //\text{END of proof of } \mathcal{T} \vdash A \to D_i \\
& //\text{BEGIN of proof of } \mathcal{T} \vdash A \to D_j \\
& \vdots \\
& D_n = A \to \overbrace{(D_i \to B)}^{D_j} \\
& //\text{END of proof of } \mathcal{T} \vdash A \to D_j \\
\text{(C2) } B := D_i, C := B : \quad & D_{n+1} = (A \to (D_i \to B)) \to ((A \to D_i) \to (A \to B)) \\
\text{MP}(D_n, D_{n+1}) : \quad & D_{n+2} = (A \to D_i) \to (A \to B) \\
\text{MP}(D_k, D_{n+2}) : \quad & D_{n+3} = A \to B
\end{aligned}$$

Corollary Cor2

$A \vdash A \lor B$ for all A, B

$$\begin{aligned}
& A \vdash \neg A \to B = A \lor B \\
& \Updownarrow \text{(DT)} \\
\text{ALL}(A) : \quad & \{A, \neg A\} \vdash B
\end{aligned}$$

$\Rightarrow$ we can add a deduction rule $\dfrac{A}{A \lor B}$ (and $\dfrac{B}{A \lor B}$ was already proved in Cl9)

Corollary Cor3

$A \vdash \neg\neg A, \qquad \vdash A \to \neg\neg A$ for all A

$$\begin{aligned}
& \vdash A \to \overbrace{(\neg A \to \mathbf{0})}^{\neg\neg A} \\
& \Updownarrow \text{(DT)} \\
& A \vdash \neg A \to \mathbf{0} \\
& \Updownarrow \text{(DT)} \\
\text{ALL}(A) : \quad & \{A, \neg A\} \vdash \mathbf{0}
\end{aligned}$$

Corollary Cor4

$\neg\neg A \vdash A, \qquad \vdash \neg\neg A \to A$ for all A

$$\begin{aligned}
\text{Cor3, } A := \neg A : \quad & D_1 = \neg A \to \neg\neg\neg A \\
\text{(C3) } B := \neg\neg A : \quad & D_2 := (\neg A \to \neg\neg\neg A) \to (\neg\neg A \to A) \\
\text{MP}(D_1, D_2) : \quad & D_3 = \neg\neg A \to A
\end{aligned}$$

Corollary Cor5

$\vdash A \leftrightarrow \neg\neg A$ (can be added to axioms)

$\vdots$ *(long way)*

Theorem
$\{A, B\} \vdash A \wedge B$,
$A \wedge B \vdash A$,
$A \wedge B \vdash B$.

Corollary
$\{A \to B, B \to A\} \vdash A \leftrightarrow B$,
$A \leftrightarrow B \vdash A \to B$,
$A \leftrightarrow B \vdash B \to A$.

Relation

$$A \approx B \text{ iff } \vdash A \leftrightarrow B$$

is an equivalence.

How can we simplify the proofs?

$$\begin{array}{lr}
\text{(TI)}: & \vdash (A \to C) \to ((C \to B) \to (A \to B)) \\
& \Updownarrow \text{(DT)} \\
& A \to C \vdash (C \to B) \to (A \to B) \\
& \Updownarrow \text{(DT)} \\
& \{C \to B, A \to C\} \vdash A \to B \\
& \Updownarrow \text{(DT)} \\
& C \to B \vdash (A \to C) \to (A \to B) \\
B :=: C: & B \to C \vdash (A \to B) \to (A \to C) \\
& B \leftrightarrow C \vdash (A \to B) \leftrightarrow (A \to C) \\
& B \approx C \Rightarrow (A \to B) \approx (A \to C)
\end{array}$$

$$\begin{array}{lr}
\text{(TI)}: & \vdash (C \to B) \to ((B \to A) \to (C \to A)) \\
& \Updownarrow \text{(DT)} \\
& C \to B \vdash (B \to A) \to (C \to A) \\
B :=: C: & B \to C \vdash (C \to A) \to (B \to A) \\
& B \leftrightarrow C \vdash (B \to A) \leftrightarrow (C \to A) \\
& B \approx C \Rightarrow (B \to A) \approx (C \to A)
\end{array}$$

Substitutions by equivalent formulas can be applied locally to any subformulas.

INTERPLAY OF SYNTAX AND SEMANTICS OF CLASSICAL LOGIC

Weak soundness
Each provable formula is a tautology, i.e., if $\vdash A$, then $\models A$.
Strong soundness
For any theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.
Weak completeness
Each tautology is provable, i.e., if $\models A$, then $\vdash A$.
Strong completeness
For any finite theory $\mathcal{T}$, if $\mathcal{T} \models A$, then $\mathcal{T} \vdash A$.

BASIC LOGIC (BL)

AS AN EXAMPLE OF A MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX OF BASIC LOGIC 1

$\mathcal{A}$ ... countable set of propositional variables
$\mathcal{L} = \{\rightarrow, \mathbf{0}, \wedge\}$ ... the set of logical connectives:
$\rightarrow$ ... (binary) implication
$\mathbf{0}$ ... (nulary) false
$\wedge$ ... (binary) conjunction (NEW)
Formulas constructed as usual
Derived connectives:
$\neg A = A \rightarrow \mathbf{0}$ ... (unary) negation
$\mathbf{1} = \neg\mathbf{0} = \mathbf{0} \rightarrow \mathbf{0}$ ... (nulary) true
$A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A)$ ... (binary) equivalence
$A \underset{\mathrm{S}}{\wedge} B = A \wedge (A \rightarrow B)$
$A \overset{\mathrm{S}}{\vee} B = ((A \rightarrow B) \rightarrow B) \underset{\mathrm{S}}{\wedge} ((B \rightarrow A) \rightarrow A)$
no $A \vee B$ in general

SEMANTICS OF BASIC LOGIC

In general: a BL-algebra, here only

Standard semantics

the set of truth values ... $[0, 1]$
$\wedge$ ... continuous fuzzy conjunction $\underset{\cdot}{\wedge}$
$\rightarrow$ ... residuum $\underset{\cdot}{\rightarrow}$ of $\underset{\cdot}{\wedge}$
$\mathbf{0}$ ... 0
Even the standard semantics is not unique, it depends on the choice of the continuous fuzzy conjunction.
Interpretation of derived connectives:
$\neg$ ... $\underset{\cdot}{\neg}$, where $\underset{\cdot}{\neg}\alpha = \alpha \underset{\cdot}{\rightarrow} 0$

$\mathbf{1}$ ... 1
$\leftrightarrow$ ... $\underset{\cdot}{\leftrightarrow}$, where $\alpha \underset{\cdot}{\leftrightarrow} \beta = (\alpha \underset{\cdot}{\rightarrow} \beta) \underset{\cdot}{\wedge} (\beta \underset{\cdot}{\rightarrow} \alpha)$

$\underset{\mathrm{S}}{\wedge}$ ... $\underset{\mathrm{S}}{\wedge} = \min$
$\overset{\mathrm{S}}{\vee}$ ... $\overset{\mathrm{S}}{\vee} = \max$

Exercise

Verify that the interpretation of $\underset{\mathrm{S}}{\wedge}, \overset{\mathrm{S}}{\vee}$ is independent of the choice of the fuzzy conjunction.

An **evaluation** (**truth assignment**) can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.
Conjunction $\wedge$ is introduced separately, as its semantics cannot be derived from the implication (as an expression using the other operations).
There are different generalizations of the notion of tautology:
1-tautology is a formula A which is always evaluated to 1 (by all possible evaluations with values with any BL-algebra, in particular, for any continuous fuzzy conjunction as an interpretation of $\wedge$ and its residuum as an interpretation of $\rightarrow$)
Notation: $\models A$
Moreover, for any theory $\mathcal{T}$,
$\mathcal{T} \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in \mathcal{T} : e(B) = 1$.

SYNTAX OF BASIC LOGIC 2

Logical axioms

$$
\begin{array}{rl}
\text{(A1)} & (A \to B) \to ((B \to C) \to (A \to C)) \\
\text{(A2)} & A \wedge B \to A \\
\text{(A3)} & A \wedge B \to B \wedge A \\
\text{(A4)} & A \wedge (A \to B) \to B \wedge (B \to A) \\
\text{(A5a)} & (A \to (B \to C)) \to (A \wedge B \to C) \\
\text{(A5b)} & (A \wedge B \to C) \to (A \to (B \to C)) \\
\text{(A6)} & ((A \to B) \to C) \to (((B \to A) \to C) \to C) \\
\text{(A7)} & \mathbf{0} \to A
\end{array}
$$

Deduction rule: Modus Ponens

$\mathrm{MP}(A,\ A \to B): \quad \dfrac{A,\ A \to B}{B}$

Theory = set of formulas (**special axioms**)
Proofs and **provable formulas** (=**theorems**) are defined as usual
Notation: $\vdash A, \qquad \mathcal{T} \vdash A$

Example 1

(C1) $\quad A \to (B \to A) \qquad$ is provable in BL:

$$
\begin{array}{rl}
\text{(A2)}: & D_1 = A \wedge B \to A \\
\text{(A5b)}, C := A: & D_2 = (A \wedge B \to A) \to (A \to (B \to A)) \\
\mathrm{MP}(D_1, D_2): & D_3 = A \to (B \to A)
\end{array}
$$

$\Rightarrow$ (C1) can be added to axioms of BL

Proposition 1

Consequence of (A1):

$$\{A \to B,\ B \to C\} \vdash A \to C$$

$\Rightarrow$ we can add a deduction rule

$\mathrm{TI}(A \to B,\ B \to C)\colon \ \dfrac{A \to B,\ B \to C}{A \to C} \qquad$ (transitivity of implication)

Example 2

$\vdash (A \to (B \to C)) \to (B \to (A \to C))$
(**Exchange rule**, also called "exchange axiom")

$$
\begin{array}{rl}
\text{(A1) } A := B \wedge A, & \\
B := A \wedge B: & D_1 = (B \wedge A \to A \wedge B) \to ((A \wedge B \to C) \to (B \wedge A \to C)) \\
\text{(A3) } A :=: B: & D_2 = B \wedge A \to A \wedge B \\
\mathrm{MP}(D_2, D_3): & D_3 = (A \wedge B \to C) \to (B \wedge A \to C) \\
\text{(A5a)}: & D_4 = (A \to (B \to C)) \to (A \wedge B \to C) \\
\text{(A5b) } A :=: B: & D_5 = (B \wedge A \to C) \to (B \to (A \to C)) \\
\mathrm{TI}(D_4, D_3): & D_6 = (A \to (B \to C)) \to (B \wedge A \to C) \\
\mathrm{TI}(D_6, D_5): & D_7 = (A \to (B \to C)) \to (B \to (A \to C))
\end{array}
$$

Example 3

$\vdash A \to A$
For brevity, let B denote a provable formula, e.g., axiom (A1).

$$\begin{array}{rl} \text{(A1)}: & D_1 = B \\ \text{Ex. } 2, C := A: & D_2 = (A \to (B \to A)) \to (B \to (A \to A)) \\ \text{(C1)}: & D_3 = A \to (B \to A) \\ \text{MP}(D_3, D_2): & D_4 = B \to (A \to A) \\ \text{MP}(D_1, D_4): & D_5 = A \to A \end{array}$$

From the deduction theorem in classical logic, only one direction holds in basic logic:
Theorem
$\mathcal{T}$ ... theory
A, B ... formulas
$\mathcal{T} \cup \{A\} \vdash B \Longleftarrow \mathcal{T} \vdash A \to B$
The proof is the same as in classical logic.

The other direction requires a weakening:
Theorem
$\mathcal{T}$ ... theory
A, B ... formulas
$\mathcal{T} \cup \{A\} \vdash B \Rightarrow \exists n \in N : (\mathcal{T} \vdash A^n \to B)$,
where $A^n = \underbrace{(A \wedge (A \wedge \cdots (A \wedge A) \cdots))}_{n\times}$
In view of (A5a), (A5b),
$(A^n \to B) \leftrightarrow \underbrace{(A \to (A \to \cdots (A \to}_{n\times} B) \cdots))$

Proof
Proof by contradiction: Suppose that there is a formula B such that $\mathcal{T} \cup \{A\} \vdash B$ and $\forall n \in N : (\mathcal{T} \not\vdash A^n \to B)$.

1. B is neither an axiom, nor a special axiom ($\in \mathcal{T}$) because then $\mathcal{T} \vdash B$,

$$\begin{array}{rl} & D_1 = B \\ RI(D_1): & D_2 = A^n \to B \end{array}$$

hence $\mathcal{T} \vdash A^n \to B$.

2. $B \neq A$ because $\vdash A^n \to A$ (without the need of $\mathcal{T}$):

$$\begin{array}{lrl} \text{for } n = 1: & & \vdash A \to A \\ \text{for } n > 1: & \text{(A2)}: & \vdash \underbrace{A \wedge A^{n-1}}_{A^n} \to A \end{array}$$

3. B is obtained by deduction in the proof of $\mathcal{T} \cup \{A\} \vdash B$.
WLOG, we choose for B a formula with the shortest possible proof; its shortest proof must be of the following form:

$$\begin{array}{rl} & \vdots \\ & D_i \\ & \vdots \\ & D_j = D_i \to B \\ & \vdots \\ \text{MP}(D_i, D_j): & D_m = B \end{array}$$

for $i < j < m$ or $j < i < m$.

The proofs of $\mathcal{T} \cup \{A\} \vdash D_i$, $\mathcal{T} \cup \{A\} \vdash D_j$ are of lengths $< m$, therefore there are $n, k \in \mathbb{N}$ such that

$$\mathcal{T} \vdash A^n \to D_i$$
$$\mathcal{T} \vdash A^k \to D_j = A^k \to (D_i \to B)$$

We shall prove $\mathcal{T} \vdash A^{n+k} \to B$:

//BEGIN of proof of $\mathcal{T} \vdash A^n \to D_i$
$\vdots$
$D_k = A^n \to D_i$
//END of proof of $\mathcal{T} \vdash A^n \to D_i$
//BEGIN of proof of $\mathcal{T} \vdash A^k \to D_j$
$\vdots$
$D_n = A^k \to \overbrace{(D_i \to B)}^{D_j}$
//END of proof of $\mathcal{T} \vdash A^k \to D_j$

Here, in classical logic, we used (C2), which is not valid in basic logic.
We can use exchange axiom (EA) and transitivity of implication (TI):

$$\begin{aligned} \text{EA}(D_n): &\quad D_{n+1} = D_i \to (A^k \to B) \\ \text{TI}(D_k, D_{n+1}): &\quad D_{n+2} = A^n \to (A^k \to B) \\ \text{(A5a)}(D_{n+2}): &\quad D_{n+3} = A^{n+k} \to B \end{aligned}$$

Deduction theorem in basic logic

$\mathcal{T}$ ... theory
A, B ... formulas
$\mathcal{T} \cup \{A\} \vdash B$ iff $\exists n \in N : (\mathcal{T} \vdash A^n \to B)$,
where $A^n = \underbrace{(A \wedge (A \wedge \cdots (A \wedge A) \cdots))}_{n\times}$

Example 4

$\vdash A \to (B \to A \wedge B)$

$$\begin{aligned} \text{(A3)}: &\quad D_1 = A \wedge B \to A \wedge B \\ \text{(A5b)}, C := A \wedge B: &\quad D_2 = (A \wedge B \to A \wedge B) \to (A \to (B \to A \wedge B)) \\ \text{MP}(D_1, D_2): &\quad D_3 = A \to (B \to A \wedge B) \end{aligned}$$

Corollary of the deduction theorem

$A \vdash B \to A \wedge B$
$\{A, B\} \vdash A \wedge B$

$$\begin{aligned} \text{(A2)}: &\quad A \wedge B \vdash A \\ \&\text{(A3)}: &\quad A \wedge B \vdash B \end{aligned}$$

Corollary

$\{A \to B, B \to A\} \vdash A \leftrightarrow B$,

$A \leftrightarrow B \vdash A \to B\,,$
$A \leftrightarrow B \vdash B \to A\,.$
Relation $A \approx B$ iff $\vdash A \leftrightarrow B$ is an equivalence and $A \wedge B \approx B \wedge A\,.$

Example of deduction

$A \vdash B \to A \wedge (A \wedge B)$

$$\begin{aligned}
\text{SA}: &\quad D_1 = A\\
\text{Ex. 4, } B := A \wedge B: &\quad D_2 = A \to (A \wedge B \to A \wedge (A \wedge B))\\
\text{MP}(D_1, D_2): &\quad D_3 = A \wedge B \to A \wedge (A \wedge B)\\
\text{(A5b)},\\
C := A \wedge (A \wedge B): &\quad D_4 = (A \wedge B \to A \wedge (A \wedge B)) \to (A \to (B \to A \wedge (A \wedge B)))\\
\text{MP}(D_3, D_4): &\quad D_5 = A \to (B \to A \wedge (A \wedge B))\\
\text{MP}(D_1, D_5): &\quad D_6 = B \to A \wedge (A \wedge B)
\end{aligned}$$

Here $\not\vdash A \to (B \to A \wedge (A \wedge B))$
but $\vdash A \wedge A \to (B \to A \wedge (A \wedge B))$
$\vdash A \to (A \to (B \to A \wedge (A \wedge B)))$
$\vdash A \to (A \wedge B \to A \wedge (A \wedge B))$
(by substitution $B := A \wedge B$ in Example 4)

How can we simplify the proofs?

$$\begin{aligned}
\text{(A1)}: &\quad \vdash (A \to C) \to ((C \to B) \to (A \to B))\\
\text{Exchange rule:} &\quad \vdash (C \to B) \to ((A \to C) \to (A \to B))\\
&\quad \Downarrow \text{(DT)}\\
&\quad C \to B \vdash (A \to C) \to (A \to B)\\
B :=: C: &\quad B \to C \vdash (A \to B) \to (A \to C)\\
&\quad B \leftrightarrow C \vdash (A \to B) \leftrightarrow (A \to C)\\
&\quad B \approx C \Rightarrow (A \to B) \approx (A \to C)
\end{aligned}$$

$$\begin{aligned}
\text{(A1)}: &\quad \vdash (C \to B) \to ((B \to A) \to (C \to A))\\
&\quad \Downarrow \text{(DT)}\\
&\quad C \to B \vdash (B \to A) \to (C \to A)\\
B :=: C: &\quad B \to C \vdash (C \to A) \to (B \to A)\\
&\quad B \leftrightarrow C \vdash (B \to A) \leftrightarrow (C \to A)\\
&\quad B \approx C \Rightarrow (B \to A) \approx (C \to A)
\end{aligned}$$

Theorem

$C \to B \vdash (C \wedge A) \to (B \wedge A)$

$$\begin{aligned}
&\quad D_1 = C \to B\\
\text{Ex. 4}: &\quad D_2 = B \to (A \to B \wedge A)\\
\text{(A1)}: &\quad D_3 = (C \to B) \to ((B \to (A \to B \wedge A)) \to (C \to (A \to B \wedge A)))\\
\text{MP}(D_1, D_3): &\quad D_4 = (B \to (A \to B \wedge A)) \to (C \to (A \to B \wedge A))\\
\text{MP}(D_2, D_4): &\quad D_5 = C \to (A \to B \wedge A)\\
\text{(A5a)}: &\quad D_6 = (C \to (A \to B \wedge A)) \to (C \wedge A \to B \wedge A)\\
\text{MP}(D_5, D_6): &\quad D_7 = C \wedge A \to B \wedge A
\end{aligned}$$

$$
\begin{aligned}
B :=: C : \quad & B \to C \vdash (B \wedge A) \to (C \wedge A) \\
& B \leftrightarrow C \vdash (B \wedge A) \leftrightarrow (C \wedge A) \\
& B \approx C \Rightarrow (B \wedge A) \approx (C \wedge A) \\
\&(\mathrm{A3}) : \quad & B \approx C \Rightarrow (A \wedge B) \approx (A \wedge C)
\end{aligned}
$$

Substitutions by equivalent formulas can be applied locally to any subformulas.

INTERPLAY OF SYNTAX AND SEMANTICS OF BASIC LOGIC

Soundness
Each provable formula is a 1-tautology, i.e., if $\vdash A$, then $\models A$.
Moreover, for any theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

Weak completeness
Each 1-tautology is provable, i.e., if $\models A$, then $\vdash A$.

Strong completeness
[Hájek 1998]
For any finite theory $\mathcal{T}$, if $\mathcal{T} \models A$, then $\mathcal{T} \vdash A$.
(We consider all evaluations with values in BL-algebras.)

Standard completeness
[Cignoli, R., Esteva, F., Godo, L., Torrens, A. 2000]
Each formula which is evaluated to 1 by all **standard** evaluations (with values in $[0, 1]$ and an arbitrary continuous fuzzy conjunction) is provable.

Example
$\not\vdash (A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
Not valid for Łukasiewicz operations and A, B, C evaluated to $1/2$.

Exercise
Which axioms of the classical logic are 1-tautologies of BL (and hence provable in BL)?

How do the properties of interpretation of conjunction (fuzzy conjunction) follow from logical axioms?

Commutativity of $\underset{\cdot}{\wedge}$ follows directly from (A3).

Boundary condition:
Example 4, $A := \mathbf{1}$: $\quad \mathbf{1} \to (B \to \mathbf{1} \wedge B)$
MP: $\quad B \to \mathbf{1} \wedge B$
the reverse implication follows from (A2)

Monotonicity of $\underset{\cdot}{\wedge}$ as a crisp property means
$(\vdash B \to C) \Rightarrow (\vdash B \wedge A \to C \wedge A)$

We already proved monotonicity of $\underset{\cdot}{\wedge}$ as a fuzzy property (which is stronger):
$\vdash (B \to C) \to (B \wedge A \to C \wedge A)$

Associativity of $\underset{\cdot}{\wedge}$:
TFAE (A5):
$(A \wedge B) \wedge C \to D$
$A \wedge B \to (C \to D)$
$A \to (B \to (C \to D))$
Now we use the equivalence of subformulas:
$B \to (C \to D)$

$B \wedge C \to D$
$A \to (B \wedge C \to D)$
$A \wedge (B \wedge C) \to D$

We proved:
$((A \wedge B) \wedge C \to D) \leftrightarrow (A \wedge (B \wedge C) \to D)$
for all D, in particular, for $D := (A \wedge B) \wedge C$:
$((A \wedge B) \wedge C \to (A \wedge B) \wedge C) \leftrightarrow$
$(A \wedge (B \wedge C) \to (A \wedge B) \wedge C)$
MP: $A \wedge (B \wedge C) \to (A \wedge B) \wedge C$
and for $D := A \wedge (B \wedge C)$ the other implication, hence
$A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$

GÖDEL LOGIC

Standard semantics:
$\wedge$ ... idempotent=standard fuzzy conjunction, $\underset{S}{\wedge} = \min$
$\to$ ... residuum of $\underset{S}{\wedge}$, Gödel implication $\underset{S}{\to}$
$\neg$ ... Gödel generalized negation $\underset{G}{\neg}$:

$$\underset{G}{\neg} x = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise} \end{cases}$$

There is no connective in Gödel logic which is interpreted by the standard fuzzy negation.
Syntax:
Axioms (A1)–(A7) and
(G) $A \to A \wedge A$
Corollary: $A \leftrightarrow A \wedge A$
Deduction rule: Modus Ponens

Only in Gödel logic the classical deduction theorem holds (because $A^n \leftrightarrow A$):
Deduction theorem in Gödel logic

$\mathcal{T}$ ... a finite theory
A, B ... formulas
$\mathcal{T} \cup \{A\} \vdash B$ iff $\mathcal{T} \vdash A \to B$

Standard completeness of Gödel logic

$(\vdash A) \iff (\models A)$ (i.e., theorems are exactly 1-tautologies w.r.t. $[0,1]$ with Gödel operations).
Moreover, for any finite theory $\mathcal{T}$, $(\mathcal{T} \vdash A) \iff (\mathcal{T} \models A)$.

$$\begin{array}{ccc} \text{theorem of CL} & \iff & \text{tautology of CL} \\ \Uparrow & & \Uparrow \\ \text{theorem of GL} & \iff & \text{1-tautology of GL} \\ \Uparrow & & \Uparrow \\ \text{theorem of BL} & \iff & \text{1-tautology of BL} \end{array}$$

PRODUCT LOGIC

Standard semantics:
$\wedge$ ... product (or any strict) fuzzy conjunction $\underset{P}{\wedge} = \cdot$
$\to$ ... residuum of $\underset{P}{\wedge}$, Goguen implication $\underset{P}{\to}$
$\neg$ ... Gödel negation $\underset{G}{\neg}$

There is no connective in product logic which is interpreted by the standard fuzzy negation.
Syntax:
Axioms (A1)–(A7) and
(P1) $\neg\neg C \to ((A \wedge C \to B \wedge C) \to (A \to B))$
(P2) $A \wedge (A \to \neg A) \to \mathbf{0}$
Remark: $e(\neg\neg C) = 1$ iff $e(C) \neq 0$ and
$\neg\neg C \vdash (A \wedge C \to B \wedge C) \leftrightarrow (A \to B)$
Alternatively, we may use a single axiom [Cintula] instead of (P1),(P2):
(P) $\neg\neg A \to ((A \to A \wedge B) \to B \wedge \neg\neg B)$
Deduction rule: Modus Ponens
Standard completeness of product logic

$(\vdash A) \iff (\models A)$ (i.e., theorems are exactly 1-tautologies w.r.t. $[0,1]$ with product operations).
Moreover, for any finite theory $\mathcal{T}$, $(\mathcal{T} \vdash A) \iff (\mathcal{T} \models A)$.

ŁUKASIEWICZ LOGIC

Standard semantics:
$\wedge$ ... Łukasiewicz (or any nilpotent) fuzzy conjunction $\underset{L}{\wedge}$
$\to$ ... residuum of $\underset{L}{\wedge}$, Łukasiewicz implication $\underset{L}{\to}$
$\neg$ ... standard fuzzy negation $\underset{S}{\neg}\, x = 1 - x$
Syntax:
Axioms (A1)–(A7) and
(L) $\neg\neg A \to A$
Corollary: $\neg\neg A \leftrightarrow A$
Deduction rule: Modus Ponens

Only here the negation is interpreted by the standard negation, $e(\neg A) = \underset{S}{\neg}\, e(A)$, and has usual nice properties.

Example L1: $(B \to A) \leftrightarrow (\neg A \to \neg B)$

$$\begin{aligned}
(A1): &\quad (A \to B) \to ((B \to C) \to (A \to C)) \\
(C := 0): &\quad (A \to B) \to (\neg B \to \neg A) \\
(A := \neg A, B := \neg B): &\quad \neg(A \to \neg B) \to (\underbrace{\neg\neg B}_{B} \to \underbrace{\neg\neg A}_{A})
\end{aligned}$$

Example L2:

$$(A \to B) \leftrightarrow \neg(A \wedge \neg B) \tag{1}$$

"→":

$$\begin{aligned}
(\text{Example 3}): &\quad (A \to B) \to (A \to B) \\
(\text{Exchange axiom}): &\quad A \to \underbrace{((A \to B) \to B)} \\
(\text{Example L1}): &\quad A \to \overbrace{(\neg B \to \neg(A \to B))} \\
(\text{A5a}): &\quad A \wedge \neg B \to \neg(A \to B) \\
(\text{Example L1}): &\quad (A \to B) \to \neg(A \wedge \neg B)
\end{aligned}$$

"←":

$$\begin{aligned}
(\text{Example 4}, B := \neg B): &\quad A \to \underbrace{(\neg B \to (A \wedge \neg B))} \\
(\text{Example L1}): &\quad A \to \overbrace{(\neg(A \wedge \neg B) \to B)} \\
(\text{Exchange axiom}): &\quad \neg(A \wedge \neg B) \to (A \to B)
\end{aligned}$$

Corollary:
$$A \wedge B = \neg(A \to \neg B) \tag{2}$$

Alternative axiomatization [Łukasiewicz & Tarski] with only $\to$, $\mathbf{0}$, conjunction is considered a connective **derived** by (2),
(L1) $A \to (B \to A)$
(L2) $(A \to B) \to \big((B \to C) \to (A \to C)\big)$
(L3) $(\neg A \to \neg B) \to (B \to A)$
(L4) $\big((A \to B) \to B\big) \to \big((B \to A) \to A\big)$ (**Łukasiewicz axiom**)
Verification of these axioms in (BL) with (L):
(L1)=(C1) (valid in BL, see Example 1)
(L2)=(A1)
(L3)=(C3, see Example L1)

(L4)

$$\begin{array}{rr}
(\text{A4}, A := \neg A, B := \neg B) & \neg A \wedge \underbrace{(\neg A \to \neg B)} \to \neg B \wedge \underbrace{(\neg B \to \neg A)} \\
(\text{Example L1}) & \underbrace{\neg A \wedge \overbrace{(B \to A)}} \to \underbrace{\neg B \wedge \overbrace{(B \to A)}} \\
(\text{A3}) & \overbrace{(B \to A) \wedge \neg A} \to \overbrace{(B \to A) \wedge \neg B} \\
(\text{Example L1}) & \neg\big((B \to A) \wedge \neg A\big) \to \neg\big((B \to A) \wedge \neg B\big) \\
(2) & \big((B \to A) \to A\big) \to \big((B \to A) \to B\big)
\end{array}$$

$$\begin{array}{rll}
A \overset{S}{\vee} B = & \big((A \to B) \to B\big) \underset{S}{\wedge} \big((B \to A) \to A\big) & \\
(\text{L4}) & \big((A \to B) \to B\big) \to \big((B \to A) \to A\big) & (\text{Łukasiewicz axiom}) \\
 & \big((A \to B) \to B\big) \leftrightarrow \big((B \to A) \to A\big) & \\
A \overset{S}{\vee} B = & \big((A \to B) \to B\big) & \\
e(A) \overset{S}{\vee} e(B) = & e\big((A \to B) \to B\big) &
\end{array}$$

Standard completeness of Łukasiewicz logic

$(\vdash A) \iff (\models A)$ (i.e., theorems are exactly 1-tautologies w.r.t. $[0, 1]$ with Łukasiewicz operations).
Moreover, for any finite theory $\mathcal{T}$, $(\mathcal{T} \vdash A) \iff (\mathcal{T} \models A)$.

RATIONAL PAVELKA LOGIC (RPL)

[Pavelka] Instead of $\{A_1, \ldots, A_k\} \vdash D_n$:

$$\begin{array}{c} D_1 \\ D_2 \\ \vdots \\ D_n \end{array}$$

we want graded formulas like (A_i, r_i) $(r_i \in [0,1] \cap \mathbb{Q})$ ($\mathbb{Q}$ for countability):
$\{(A_1, r_1), \ldots, (A_k, r_k)\} \vdash (D_n, s_n)$:

$$\begin{array}{c}(D_1, s_1)\\(D_2, s_2)\\\vdots\\(D_n, s_n)\end{array}$$

Graded formulas are proved using **graded axioms** and **graded deduction rules** like **Generalized Modus Ponens** GMP : $\dfrac{(A, r),\ (A \to B, s)}{(B, r \underset{\text{L}}{\wedge} s)}$
(based on Łukasiewicz logic because of its continuous interpretation)

HÁJEK'S FORMULATION OF RATIONAL PAVELKA LOGIC

RPL as a minimal modification of Łukasiewicz logic:
Add constants $\mathbf{r}$, $r \in [0,1] \cap \mathbb{Q}$ (where $\mathbf{0}$, $\mathbf{1} = \neg\mathbf{0}$ keep the previous meaning)
Syntax:
Axioms of Łukasiewicz logic ((A1)–(A7) and (L), or (L1)–(L4)),
and **bookkeeping axioms**:
(RPL) $\quad (\mathbf{r} \to \mathbf{s}) \leftrightarrow \mathbf{t} \qquad \forall r, s, t \in [0,1] \cap \mathbb{Q} :\ r \underset{\text{L}}{\to} s = t$
Corollary for $\mathbf{s} = \mathbf{0}$: $\quad \neg\mathbf{r} \leftrightarrow \mathbf{t} \qquad \forall r, t \in [0,1] \cap \mathbb{Q} :\ \underset{\text{S}}{\neg} r = t$

Graded formula (A, r) is equivalent to $(\mathbf{r} \to A)$.
Ordinary formula A is equivalent to $(A, 1)$, i.e., $(\mathbf{1} \to A)$.

Deduction rule: Modus Ponens
Generalized Modus Ponens is obtained as a special case of Modus Ponens of Łukasiewicz logic:

$$\frac{\mathbf{r} \to A,\ \mathbf{s} \to (A \to B)}{\mathbf{t} \to B} \qquad \forall r, s, t \in [0,1] \cap \mathbb{Q} :\ t = r \underset{\text{L}}{\wedge} s$$

$$\begin{aligned}
\text{SA} :&\quad D_1 = \mathbf{r} \to A\\
\text{SA} :&\quad D_2 = \mathbf{s} \to (A \to B)\\
\text{Exchange rule for } D_2 :&\quad D_3 = A \to (\mathbf{s} \to B)\\
\text{TI}(D_1, D_3) :&\quad D_4 = \mathbf{r} \to (\mathbf{s} \to B)\\
\text{(A5a)}, A := \mathbf{r}, B := \mathbf{s}, C := B :&\quad D_5 = (\mathbf{r} \to (\mathbf{s} \to B)) \to (\mathbf{r} \wedge \mathbf{s} \to B)\\
\text{MP}(D_4, D_5) :&\quad D_6 = \mathbf{r} \wedge \mathbf{s} \to B\\
D_6, \mathbf{r} \wedge \mathbf{s} \leftrightarrow \mathbf{t} :&\quad D_7 = \mathbf{t} \to B
\end{aligned}$$

The deduction theorem of Łukasiewicz logic remains valid in RPL.

Semantics:
$e(\mathbf{r}) = r \qquad \forall r \in [0,1] \cap \mathbb{Q}$
Thus $e(\mathbf{r} \to A) = 1 \iff e(A) \geq r$

Besides theorems and 1-tautologies (defined as usual),
for a formula A and theory $\mathcal{T}$, we define

- the **truth degree** $||A||_{\mathcal{T}} = \inf\{e(A) \mid \forall B \in \mathcal{T} : e(B) = 1\}$

- the **provability degree** $|A|_{\mathcal{T}} = \sup\{r \in [0,1] \cap \mathbb{Q} \mid \mathcal{T} \vdash \mathbf{r} \to A\}$

Completeness Theorem for RPL:
$|A|_{\mathcal{T}} = ||A||_{\mathcal{T}}$

Theory $\mathcal{T}$ is

- **consistent**, if $\mathcal{T} \not\vdash \mathbf{0}$,
  equivalently, if $\mathcal{T} \not\vdash \mathbf{r} \qquad \forall r < 1$;
- **complete**, if $\mathcal{T} \vdash A \to \mathbf{r}$ or $\mathcal{T} \vdash \mathbf{r} \to A \qquad \forall A\ \forall r \in [0,1] \cap \mathbb{Q}$.

Lemma: Theory $\mathcal{T}$ is inconsistent iff $\mathcal{T} \vdash A \qquad \forall A$.
Lemma: If $(\mathcal{T} \not\vdash \mathbf{r} \to A)$, then $\mathcal{T} \cup \{A \to \mathbf{r}\}$ is consistent.

Lemma: Let a theory $\mathcal{T}$ be consistent and complete. Then

- $|A|_{\mathcal{T}} = \sup\{r \in [0,1] : \mathcal{T} \vdash \mathbf{r} \to A\} = \inf\{s \in [0,1] : \mathcal{T} \vdash A \to \mathbf{s}\}$;
- $|\cdot|_{\mathcal{T}}$ comutes with logical connectives, i.e., $|A \to B|_{\mathcal{T}} = |A|_{\mathcal{T}} \underset{\text{Ł}}{\to} |B|_{\mathcal{T}}$ (in particular, $|\neg A|_{\mathcal{T}} = 1 - |A|_{\mathcal{T}}$);
- function e defined by $e(A) = |A|_{\mathcal{T}}$ is an evaluation.

COMPACTNESS OF LOGICS

Compactness Theorem I: A theory is satisfiable iff each its **finite** subset (*subtheory*) is satisfiable.
Compactness Theorem I holds in classical, basic, Gödel, Łukasiewicz, and product logic and in RPL.
In classical logic, an equivalent formulation is:
Compactness Theorem II: A formula is provable in a theory iff it is provable in some its **finite** subtheory.
Compactness Theorem II holds in classical, basic, Gödel, Łukasiewicz, and product logic, but **not** in RPL.
Example:
$\forall n \in \mathbb{N} : r_n := 1 - \frac{1}{n}, \qquad A$ formula which is not provable (e.g., a variable),
$\mathcal{T} = \{\mathbf{r}_n \to A : n \in \mathbb{N}\}$,
$|A|_{\mathcal{T}} \geq \sup\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 1$,
but A cannot be proved from any finite subtheory.

CLASSICAL PREDICATE LOGIC [5]

Instead of propositions, atomic formulas are statements about some objects.
We start from a non-empty set of **objects**.
Predicates assign truth values to objects.
Additionally, we may use *object constants* and *function symbols* defining relations between objects (here not considered).
We have **quantifiers**
$\forall x\ P(x)$ may be understood as a (possibly infinite) conjunction
$P(x_1) \wedge P(x_2) \wedge \ldots$
$\exists x\ P(x)$ may be understood as a (possibly infinite) disjunction
$P(x_1) \vee P(x_2) \vee \ldots$

SYNTAX OF CLASSICAL PREDICATE LOGIC

Logical symbols:
$\mathcal{A} = \{x, y, \ldots\}$ ... countable set of *object* variables
$\mathcal{L} = \{\to, \mathbf{0}\}$ ... the set of logical connectives
$\forall, \exists$ ... quantifiers, where $\exists$ is considered *derived* by $\exists x\ A = \neg(\forall x\ \neg A)$
Special symbols:
$\mathcal{P}$ ... nonempty set of **predicates** (with assigned arity)
event. *function symbols* (here not considered)
event. *object constants* (may be substituted by nulary predicates; here not considered)
event. a special **equality** predicate (in infix notation, $x = y$), requiring additional axioms of equality (here not considered)

Formulas
• $P(x_1, \dots, x_n)$, where P is a predicate of arity n and $x_1, \dots, x_n$ are variables
• **0**
• $A \to B$, where A, B are formulas
• $\forall x\ A$, where A is a formula
• $\exists x\ A$, where A is a formula
Occurence of a variable can be
• **bound** when it is quantified,
• **free** otherwise.
Examples:
$\forall x\ Q(x, y)$ ... x is bound, y is free
$x \wedge \forall x\ P(x)$ ... the first occurrence of x is free, the rest bound
A formula can be
• **closed (=sentence)**: without free variables
• **open**: without bound variables
Formulas differing only by the notation of bound variables are considered equal.

SEMANTICS OF CLASSICAL PREDICATE LOGIC

Interpretation M:
• D_M ... nonempty set of objects (**domain, universe**; event. may be distinguished for different variables)
• interpretation of a predicate of arity n ... an n-ary relation $D_M^n \to \{0, 1\}$
Evaluation (in interpretation M): $e_M : \mathcal{A} \to D_M$ extends uniquely to all formulas.
New rules:
$(e_M(\forall x\ A) = 1) \iff e'_M(A) = 1$ for **every** evaluation $e'_M : \mathcal{A} \to D_M$ which differs from e_M at most at x
equivalently: $e_M(\forall x\ A) = \inf\{e'_M(A) : e'_M : \mathcal{A} \to D_M$ is an evaluation which differs from e_M at most at $x\}$
$(e_M(\exists x\ A) = 1) \iff e'_M(A) = 1$ for **some** evaluation $e'_M : \mathcal{A} \to D_M$ which differs from e_M at most at x
equivalently: $e_M(\exists x\ A) = \sup\{e'_M(A) : e'_M : \mathcal{A} \to D_M$ is an evaluation which differs from e_M at most at $x\}$
Notation:
$\models_M A$ ... A is true in all evaluations in interpretation M
$\models A$ ... A is true in all evaluations in **all** interpretations (**tautology**)
As before: **tautologies, contradictions, satisfiable formulas**, but only for **closed** formulas;
satisfiability of a set of **closed** formulas.
Examples of new tautologies:
$(\forall x\ P(x)) \to P(t)$
$P(t) \to (\exists x\ P(x))$
$\neg(\forall x\ P(x)) \leftrightarrow (\exists x\ \neg P(x))$
$\neg(\exists x\ P(x)) \leftrightarrow (\forall x\ \neg P(x))$
$(\forall x\ \forall y\ Q(x, y)) \leftrightarrow (\forall y\ \forall x\ Q(x, y))$
$(\exists x\ \exists y\ Q(x, y)) \leftrightarrow (\exists y\ \exists x\ Q(x, y))$
Semantical consequent: $\mathcal{T} \models A$ (without index)
only for sets of **closed** formulas
Theorem
$(\mathcal{T} \models A) \iff (\mathcal{T} \cup \{\neg A\}$ is unsatisfiable)

SYNTAX OF CLASSICAL PREDICATE LOGIC 2

Logical axioms

$$
\begin{array}{ll}
\text{(C1)} & A \to (B \to A) \\
\text{(C2)} & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\
\text{(C3)} & (\neg A \to \neg B) \to (B \to A)
\end{array}
$$

• for each formula $P(t)$ originated by substituting t for x in a formula $P(x)$:

$$(\forall 1) \qquad (\forall x\ P(x)) \to P(t)$$

• for each formula B not containing x (as a free variable):

$$(\forall 2) \qquad (\forall x\ (B \to A)) \to (B \to \forall x\ A)$$

Deduction rules

Modus Ponens: $\dfrac{A,\ A \to B}{B}$

Generalization: $\dfrac{A}{\forall x\ A}$

As before: Theory (special axioms), proof, provable formula, ...

Deduction theorem in classical predicate logic

Only when A is a **closed** formula:

$(\mathcal{T} \cup \{A\} \vdash B) \iff (\mathcal{T} \vdash A \to B)$

Proof: Analogous, only in the direction $\Rightarrow$, we must discuss also the case of a formula obtained by generalization:

4. B is obtained by generalization in the proof of $\mathcal{T} \cup \{A\} \vdash B$.
WLOG, we choose for $B = \forall x\ D_i$ a formula with the shortest possible proof; its shortest proof must be of the following form:

$$\begin{array}{l} \vdots \\ D_i \\ \vdots \\ D_m = B = \forall x\ D_i \end{array}$$

The proof of $\mathcal{T} \cup \{A\} \vdash D_i$ is of length $i < m$, therefore

$$\mathcal{T} \vdash A \to D_i$$

Proof of $\mathcal{T} \vdash A \to B$:

$$\begin{array}{rl} & \text{//BEGIN of proof of } \mathcal{T} \vdash A \to D_i \\ & \vdots \\ & D_n = A \to D_i \\ & \text{//END of proof of } \mathcal{T} \vdash A \to D_i \\ \text{Generalization}: & D_{n+1} = \forall x (A \to D_i) \\ (\forall 2): & D_{n+2} = (\forall x\ (A \to D_i)) \to (A \to \forall x\ D_i) \\ \text{MP}(D_{n_1}, D_{n+2}): & D_{n+3} = A \to \underbrace{\forall x\ D_i}_{B} \end{array}$$

Corollary

Only for **closed** formulas:

$(A \models\!\mid B) \iff (\models A \leftrightarrow B)$

It is **not** a generalization of the deduction theorem of classical propositional logic because on both sides a new deduction rule—generalization—can be used.

Theorem

$\vdash (\neg \exists x\ A) \leftrightarrow (\forall x\ \neg A)$
$\vdash (\neg \forall x\ A) \leftrightarrow (\exists x\ \neg A)$
$\vdash (\exists x\ A) \leftrightarrow \neg (\forall x\ \neg A)$
$\vdash (\forall x\ A) \leftrightarrow \neg (\exists x\ \neg A)$
Therefore we did not need axioms for $\exists$.

BASIC (FUZZY) PREDICATE LOGIC, BL$\forall$

Derived from BL as classical predicate logic from classical propositional logic.

What differs:
$\exists$ is a basic quantifier, which *cannot be derived* from $\forall$.

SEMANTICS OF BASIC PREDICATE LOGIC

Interpretation of a predicate of arity n ... an n-ary **fuzzy** relation $D_M^n \to$ **[0,1]**.
Evaluation of quantifiers in interpretation M:
$e_M(\forall x\ A) = \inf\{e'_M(A) : e'_M \colon \mathcal{A} \to D_M$ is an evaluation which differs from e_M at most at $x\}$
$e_M(\exists x\ A) = \sup\{e'_M(A) : e'_M \colon \mathcal{A} \to D_M$ is an evaluation which differs from e_M at most at $x\}$
Remark: For evaluations in more general sets of truth values than $[0, 1]$ (BL-algebras), it becomes necessary to assume the existence of $\inf, \sup$ in question.
Truth value of formula A in interpretation M: $||A||_M = \inf\{e_M(A) : e_M$ is an evaluation in $M\}$

As before:
$\models_M A$ ... A is true in all evaluations in interpretation M
$\models A$ ... A is true in all evaluations in **all** interpretations (**tautology**)

SYNTAX OF BASIC PREDICATE LOGIC

Axioms:

- (A1)–(A7)
- for each formula $P(t)$ originated by substituting t for x in a formula $P(x)$:

$$(\forall 1) \qquad (\forall x\ P(x)) \to P(t)$$
$$(\exists 1) \qquad P(t) \to (\exists x\ P(x))$$

- for each formula B not containing x (as a free variable):

$$(\forall 2) \qquad (\forall x\ (B \to A)) \to (B \to \forall x\ A)$$
$$(\exists 2) \qquad (\forall x\ (A \to B)) \to ((\exists x\ A) \to B)$$
$$(\forall 3) \qquad (\forall x\ (B \overset{\text{S}}{\vee} A)) \to ((\forall x\ A) \overset{\text{S}}{\vee} B)$$

Deduction rules:

Modus Ponens: $\dfrac{A,\ A \to B}{B}$

Generalization: $\dfrac{A}{\forall x\ A}$

Deduction theorem in basic predicate logic

(for **closed** formula A)
$(\mathcal{T} \cup \{A\} \vdash B) \iff (\exists n \in N : (\mathcal{T} \vdash A^n \to B))$

Theorem

$\vdash (\exists x\ A) \to \neg(\forall x\ \neg A)$
$\vdash (\neg\exists x\ A) \leftrightarrow (\forall x\ \neg A)$

Theorem

$e_M(A) \overset{\bullet}{\wedge} e_M(A \to B) \le e_M(B)$
in particular:
$(e_M(A) = 1) \wedge (e_M(A \to B) = 1) \Rightarrow (e_M(B) = 1)$

Corollary

$||A||_M \overset{\bullet}{\wedge} ||A \to B||_M \le ||B||_M$
in particular:
$(||A||_M = 1) \wedge (||A \to B||_M = 1) \Rightarrow (||B||_M = 1)$
$||A||_M = ||\forall x\ A||_M$
$(\models_M A) \Rightarrow (\models_M \forall x\ A)$
Interpretation M is a **model** of a theory $\mathcal{T}$ if
$||A||_M = 1$ for all $A \in \mathcal{T}$

Soundness and completeness

$(\mathcal{T} \vdash A) \iff (||A||_M = 1$ for each model M of $\mathcal{T})$
(More exactly, we have to assume evaluations with values in linearly ordered BL-algebras.)

RATIONAL PAVELKA PREDICATE LOGIC RPL∀

Derived from RPL as basic predicate logic from basic propositional logic.
What differs:

Axioms:

- (L1)–(L4)
- (RPL) (bookkeeping axioms)
- $(\forall 1), (\forall 2)$

This is sufficient because
(even in Łukasiewicz predicate logic)
$\vdash (\exists x\ A) \leftrightarrow \neg(\forall x\ \neg A)$
and also $(\forall 3)$ is provable.

Soundness and completeness

If $\mathcal{T}$ is a theory (consisting of **closed** formulas), then
$|A|_{\mathcal{T}} = ||A||_{\mathcal{T}}$,
where
$||A||_{\mathcal{T}}$ is the infimum of all evaluations over all models of $\mathcal{T}$,
$|A|_{\mathcal{T}}$ is the supremum of all provability degrees assuming $\mathcal{T}$.

[1] Cintula, P., Hájek, P., Noguera, C.: *Handbook of Mathematical Fuzzy Logic. Volume 1*, Studies in Logic 37, College Publications, 2011.

[2] Hájek, P.: *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.

[3] Klement, E.P.; Mesiar, R.; Pap, E.: *Triangular Norms.* Kluwer, Dordrecht, 2000.

[4] Navara, M., Olšák, P.: *Basics of Fuzzy Sets* (in Czech). 2nd ed. (revised), Czech Technical University, Prague, 2007.

[5] Sochor, A.: *Classical Mathematical Logic* (in Czech). Karolinum, Prague, 2001.