1 Stabilizing an inverted pendulum (cartpole problem)

Using acceleration a, we want to stabilize an inverted pendulum of efficient length ℓ. The angle φ satisfies the ODE

$$\ell\,\varphi'' - g\,\varphi + a = 0$$

where g is the acceleration of gravity, with initial conditions

$\varphi(0) = \varphi_0,\ \varphi'(0) = \omega_0.$

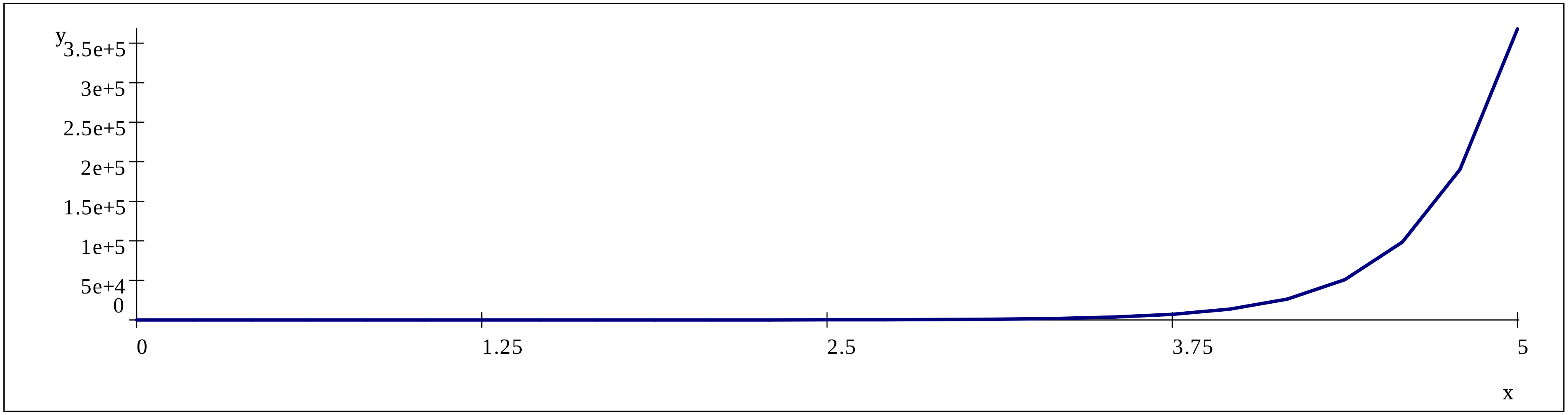
In examples, we use $g = 10,\ \ell = 1,\ \varphi_0 = 0.1,\ \omega_0 = 0.$

1.1 No controller $\quad a = 0: \quad \ell\,\varphi'' - g\,\varphi = 0$

The system

$$\varphi'' - 10\,\varphi = 0$$
$$\varphi(0) = 0.1$$
$$\varphi'(0) = 0$$

is unstable, the exact solution is: $0.05e^{t\sqrt{10}} + 0.05e^{-t\sqrt{10}}$

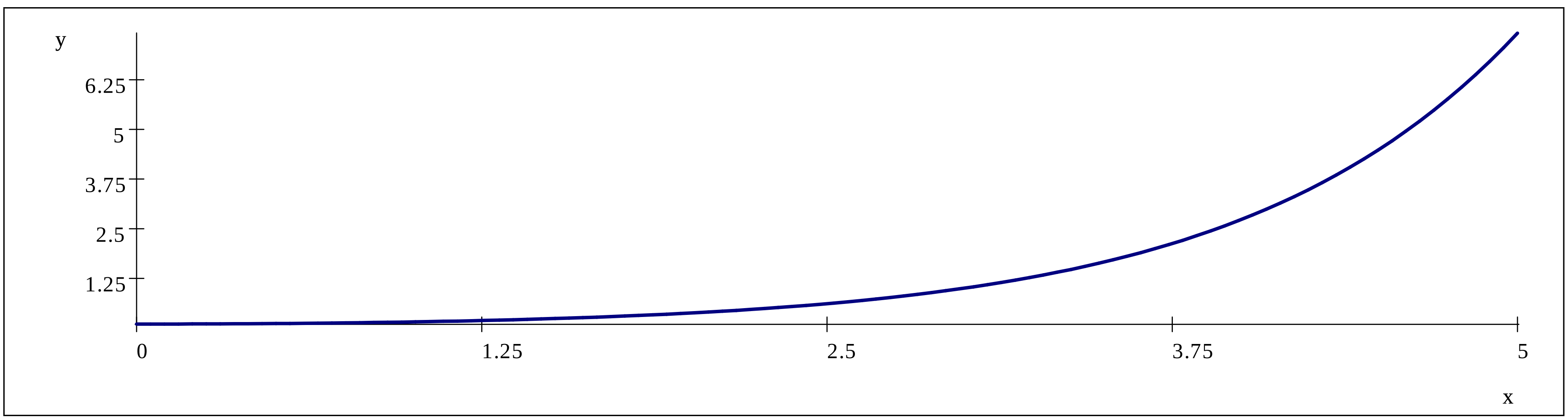


1.2 P controller $a = p\,\varphi$: $\ell\,\varphi'' - g\,\varphi + p\,\varphi = 0$

The system is unstable for all p, e.g.

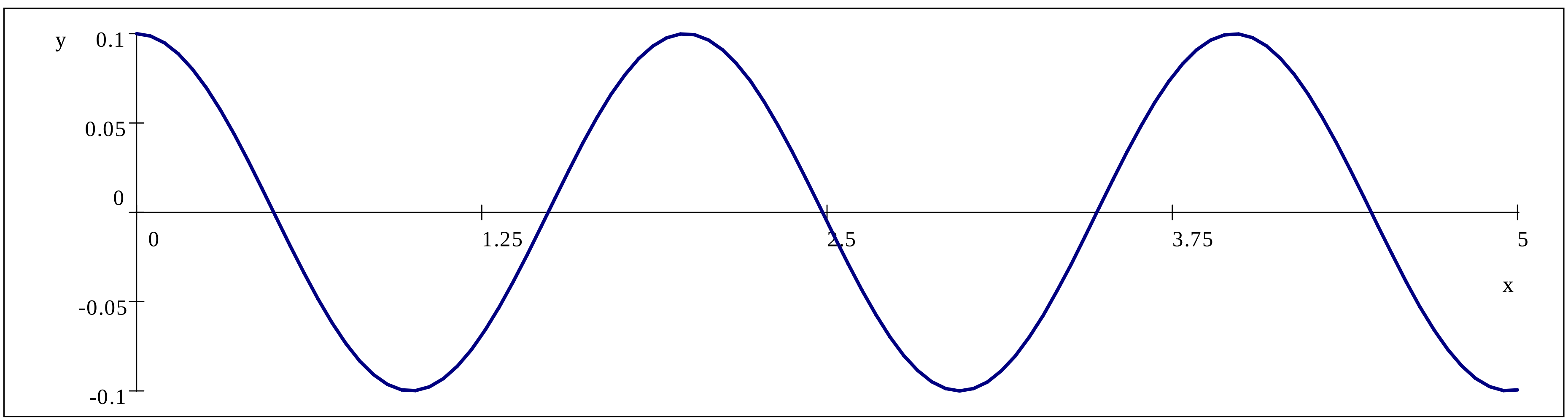
$$\varphi'' - 10\,\varphi + 9\,\varphi = 0$$
$$\varphi(0) = 0.1$$
$$\varphi'(0) = 0$$

the exact solution is: $0.05e^{t} + 0.05e^{-t}$



$$\varphi'' - 10\,\varphi + 20\,\varphi = 0$$
$$\varphi(0) = 0.1$$
$$\varphi'(0) = 0$$

the exact solution is: $0.1\cos t\sqrt{10}$



1.3 PI controller $a = p\,\varphi + i\,\int_0^t \varphi(\tau)\,d\tau$

$$\ell\,\varphi'' - g\,\varphi + p\,\varphi + i\,\int_0^t \varphi(\tau)\,d\tau = 0$$

Difficult to solve, the order increased.

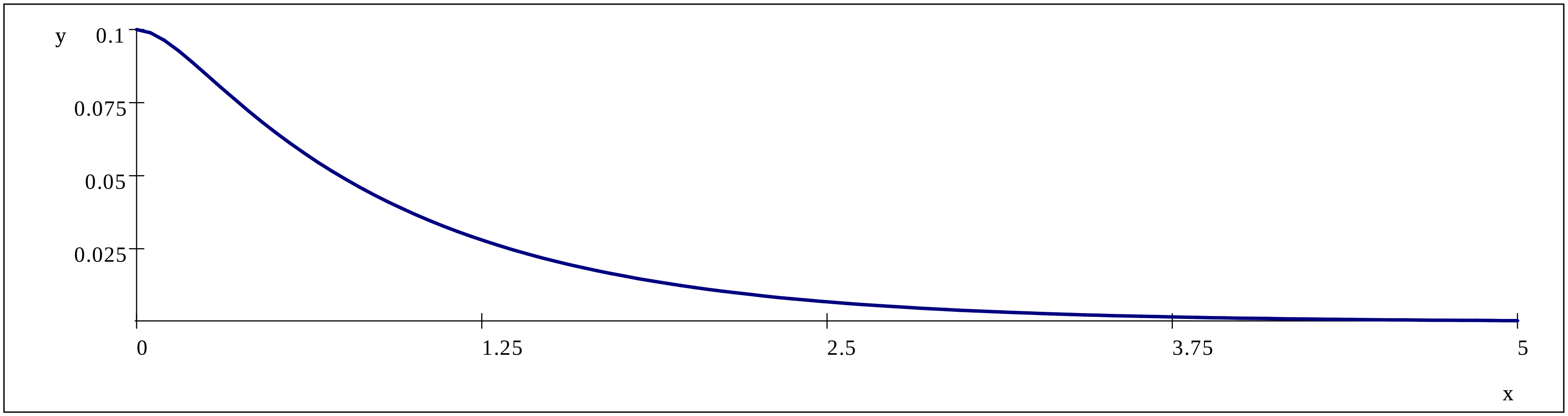
1.4 PD controller $a = p\,\varphi + d\,\varphi'$

$$\ell\,\varphi'' - g\,\varphi + p\,\varphi + d\,\varphi' = 0$$

It can be stable, e.g.

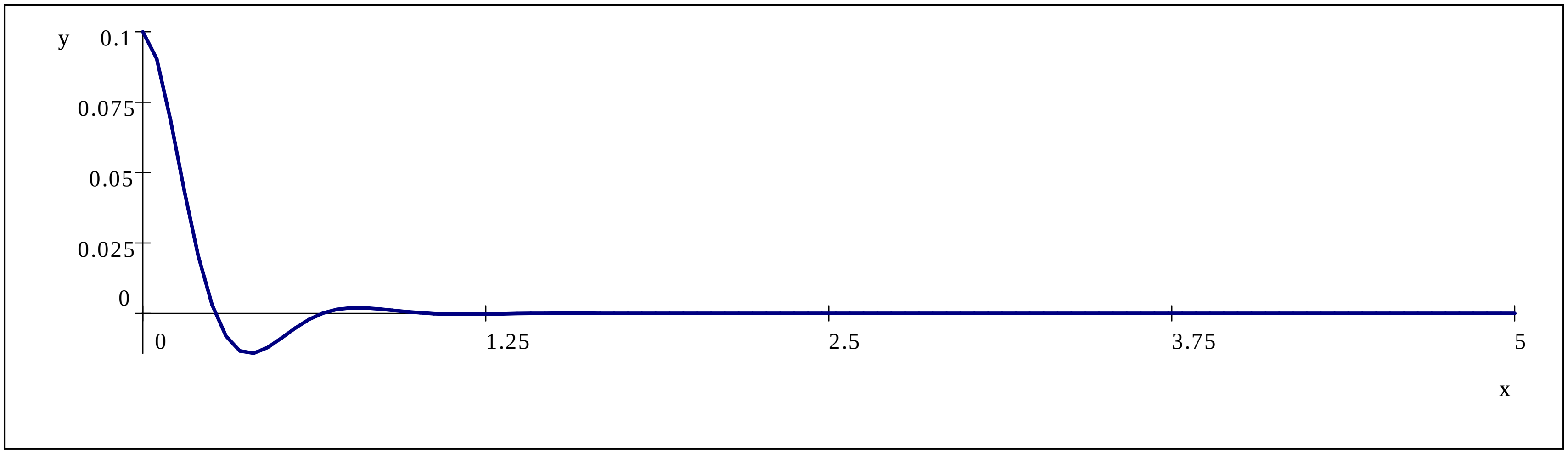
$$\begin{aligned} \varphi'' - 10\,\varphi + 20\,\varphi + 10\,\varphi' &= 0 \\ \varphi(0) &= 0.1 \\ \varphi'(0) &= 0 \end{aligned}$$

the exact solution is: $0.114\,55e^{t\left(\sqrt{15}-5\right)} - 1.455\,0 \times 10^{-2}e^{t\left(-\sqrt{15}-5\right)}$



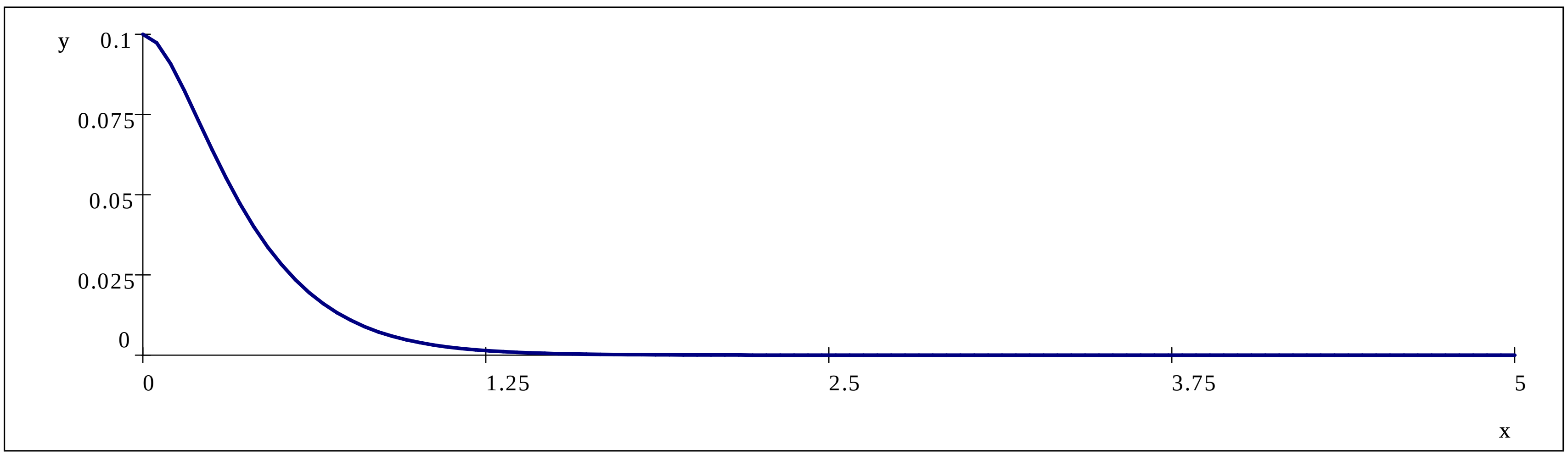
$$\varphi'' - 10\,\varphi + 100\,\varphi + 10\,\varphi' = 0$$
$$\varphi(0) = 0.1$$
$$\varphi'(0) = 0$$

the exact solution is: $0.1e^{-5t}\cos t\sqrt{65} + 6.2017 \times 10^{-2}e^{-5t}\sin t\sqrt{65}$



$$\varphi'' - 10\,\varphi + 35\,\varphi + 10\,\varphi' = 0$$
$$\varphi(0) = 0.1$$
$$\varphi'(0) = 0$$

the exact solution is: $0.1e^{-5t} + 0.5te^{-5t}$



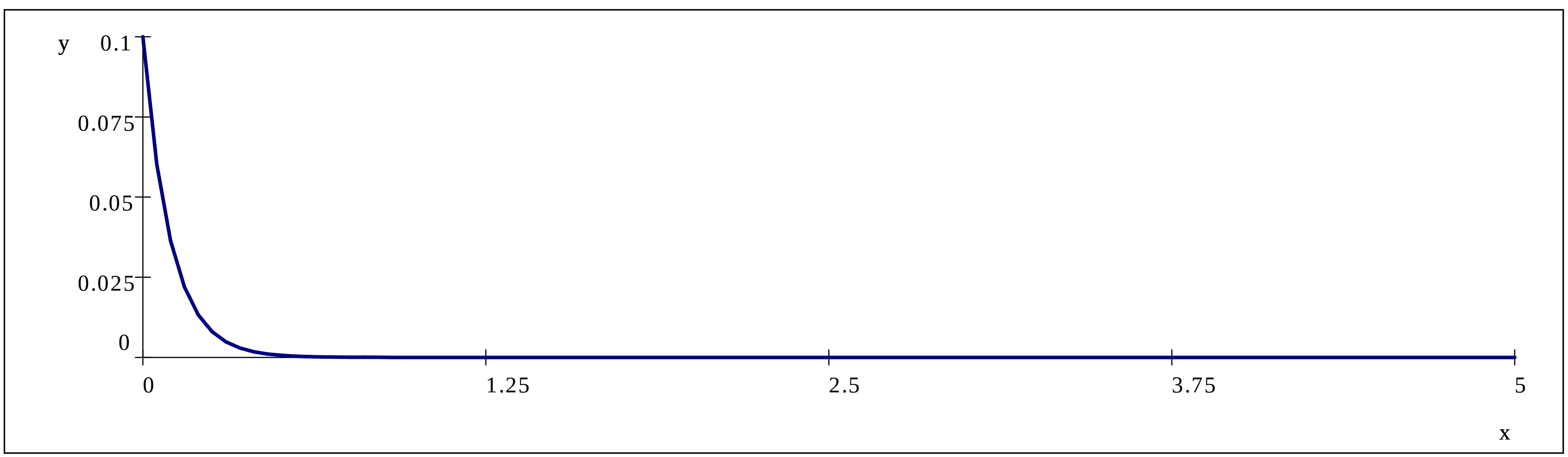
1.5 State-feedback controller $a = p\,\varphi + d\,\varphi' + c\,\varphi''$

$$\ell\,\varphi'' - g\,\varphi + p\,\varphi + d\,\varphi' + c\,\varphi'' = 0$$

It can be stable and the order decreases for $c = -\ell$ (then the second-order term vanishes), e.g.

$$\varphi'' - 10\,\varphi + 20\,\varphi + \varphi' - \varphi'' = 0$$
$$\varphi(0) = 0.1$$

the exact solution is: $0.1e^{-10t}$



2 Why these strange results?

We have to look at the characteristic values of the systems, these are roots of the denominator of the transfer function of the feedback loop,

$$G(s) = \frac{G_1(s)}{1 - G_2(s)\, G_1(s)}$$

where

$$G_1(s) = \frac{1}{g - \ell\, s^2}$$

and $G_2(s)$ is the transfer function of the controller, hence

$$G(s) = \frac{-1}{\ell\, s^2 - g + G_2(s)}$$

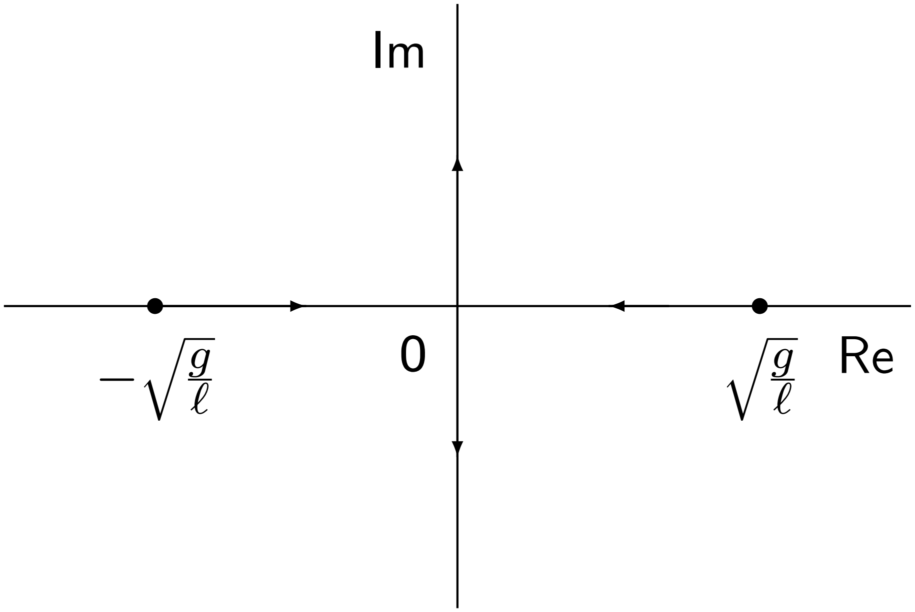
and we study the roots of $\ell\, s^2 - g + G_2(s)$

2.1 No controller $\quad G(s) = 0$: $\quad \ell s^2 - g = 0$

$s = \pm\sqrt{\frac{g}{l}}$ (in the real domain)

2.2 P controller $\quad G(s) = p$: $\quad \ell s^2 - g + p = 0$

$s = \sqrt{\frac{g-p}{l}}$ (two-valued square root in the complex domain)



The roots are symmetric w.r.t. 0, so stability cannot be achieved

2.3 PI controller $G(s) = p + \dfrac{i}{s}$

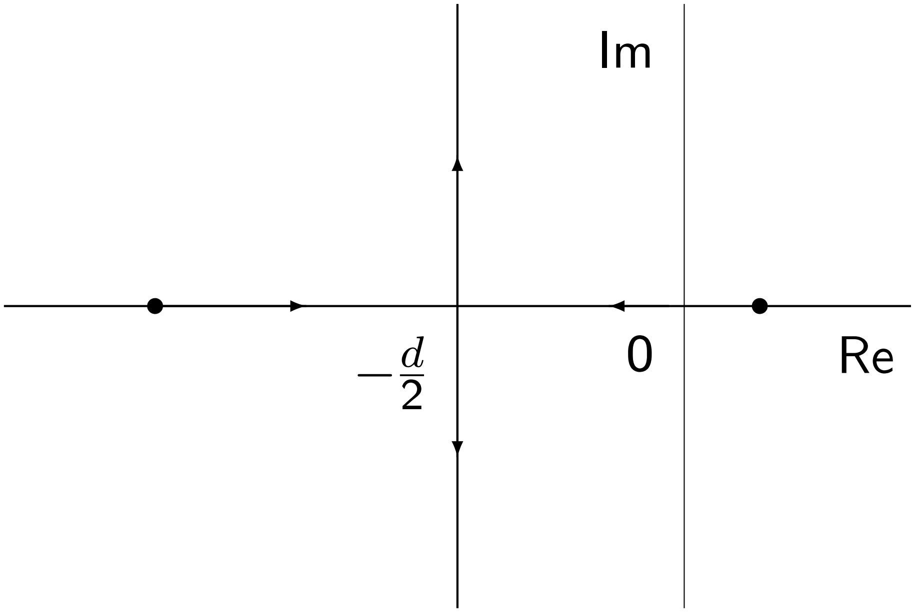
$$\ell\, s^2 - g + p + \frac{i}{s} = 0$$

Difficult to solve (3rd order, even for a drastically simplified task)

2.4 PD controller $G(s) = p + d\,s$

$$\ell\, s^2 - g + p + d\, s = 0$$

$$s = \frac{-d + \sqrt{d^2 - 4\,\ell\,(p-g)}}{2\,\ell}$$



The roots are symmetric w.r.t. $\dfrac{-d}{2\,\ell}$, so stability can be achieved

2.5 State-feedback controller $\qquad G(s) = p + d\,s + c\,s^2$

$$\ell\,s^2 - g + p + d\,s + c\,s^2 = 0$$

Any form is theoretically possible, e.g., $c = -\ell,\ p = 20,\ d = 1$:

$$\begin{aligned} s^2 - 10 + 20 + s - s^2 &= 0 \\ s &= -10 \end{aligned}$$