Center for Machine Perception presents
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Mirko Navara (Praha)
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Semantical testing of tautologies in many-valued logics
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Semantical testing of tautologies in many-valued logics

What can computers do for us?
Center for Machine Perception presents

Mirko Navara (Praha)

Semantical testing of tautologies in many-valued logics

What can computers do for us?

(And what they cannot do.)
Semantical testing of tautologies

In **Boolean algebra**: only a “small” finite number of cases, $2^n$, where $n$ is the number of different variables.
Semantical testing of tautologies

In **Boolean algebra**: only a “small” finite number of cases, $2^n$, where $n$ is the number of different variables

In **many-valued logics**: 
Semantical testing of tautologies

In **Boolean algebra**: only a “small” finite number of cases, $2^n$, where $n$ is the number of different variables

In **many-valued logics**: Depends on the choice of many-valued logic; the most interesting progress has been made in the Łukasiewicz logic, i.e., in MV-algebras
Semantical testing of tautologies in MV-algebras

It is enough to consider evaluations in
Semantical testing of tautologies in MV-algebras

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- the standard MV-algebra $[0, 1]$ [Chang 58]
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- $\{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\}, \ m \leq b_0(M)$, where $b_0(M) = 2^{(2^M)^2}$, $M$ is the number of variables [Mundici 87] developed for another reason

____________________________


1st bound

\( M \) ... the number of all occurrences of variables in the formula
\( n \) ... the number of different variables in the formula
1st bound

$M$ ... the number of all occurrences of variables in the formula

$n$ ... the number of different variables in the formula

[Mundici 87]: $m \leq b_0(M) = 2^{(2^M)^2} = 2^{4M^2}$
1st bound

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$n$ ... the number of different variables in the formula

[Mundici 87]: $m \leq b_0(M) = 2(2^M)^2 = 2^{4M^2}$

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<tr>
<th>$M$</th>
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1st bound

$M$ ... the number of all occurrences of variables in the formula

$n$ ... the number of different variables in the formula

[Mundici 87]: $m \leq b_0(M) = 2^{(2M)^2} = 2^{4M^2}$

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Complexity $\sum_{m=1}^{b_0(M)} (m + 1)^n$
1st bound

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[Mundici 87]: \( m \leq b_0(M) = 2^{(2M)^2} = 2^{4M^2} \)

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Complexity \( \sum_{m=1}^{b_0(M)} (m + 1)^n \)

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<td>( 2.361 \times 10^{21} )</td>
<td>( 1.081 \times 10^{32} )</td>
<td>( 5.575 \times 10^{42} )</td>
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"The importance of being a good teacher."
2nd bound

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[Aguzzoli, Ciabattoni, B. Gerla]: $m = b_1(M) = 2^{M-1}$
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Complexity: \( (b_1(M) + 1)^n \)

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3rd bound

[Aguzzoli, Ciabattoni, B. Gerla]: \( m \leq b(M, n) = \left\lfloor \left( \frac{M}{n} \right)^n \right\rfloor \)
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3rd bound

This approach is preferable. As a by-product, we find the minimal denominator for which the formula is not a tautology.
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*Implemented by [Brůžková 05].*
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Semantical testing of tautologies in MV-algebras 2

How do the connectives contribute to $M$ (and thus to the bounds):
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$\land$ increments $M$ by 1
Semantical testing of tautologies in MV-algebras 2

How do the connectives contribute to $M$ (and thus to the bounds):

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$\rightarrow$ increments $M$ by 1
Semantical testing of tautologies in MV-algebras 2

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$\land_s$ increments $M$ by 2 because $x \land_s y = x \land (x \rightarrow y)$
Semantical testing of tautologies in MV-algebras 2

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$\land$ increments $M$ by 2 because $\land_S x y = x \land (x \rightarrow y)$

$\lor$ increments $M$ by 2 because $\lor_S x y = (x \rightarrow y) \rightarrow y = \neg (\neg x \land \neg y)$
Semantical testing in many-valued logics 2

Related questions:
Semantical testing in many-valued logics 2

Related questions:

Semantical testing in many-valued logics 2

Related questions:

- Testing of tautologies in Gödel logic
Semantical testing in many-valued logics 2

Related questions:


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  It is enough to consider evaluations in \( \{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\} \), \( m = n + 1 \) [Baaz]
Semantical testing in many-valued logics 2

Related questions:


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Semantical testing in many-valued logics 2

Related questions:

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Semantical testing in many-valued logics 2

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- Testing of tautologies in product logic?
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...

- Testing of tautologies in basic logic?

  [Hájek; Haniková; Montagna, Pinna, and Tiezzi 03]; so far no implementation.

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Semantical testing in many-valued logics 3

Alternative approaches to testing of tautologies:
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Semantical testing in many-valued logics 3

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  Programmed by [Hähnle et al. ∼95].
- Search for counterexamples
Semantical testing in many-valued logics 4

- Search for counterexamples
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Semantical testing in many-valued logics 4

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Semantical testing in many-valued logics 4

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The latter two methods do not guarantee an ultimate answer, but they give a reasonable chance to obtain it.