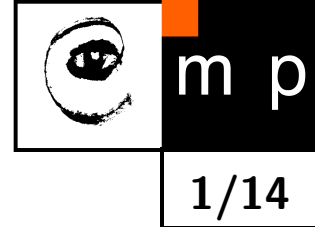
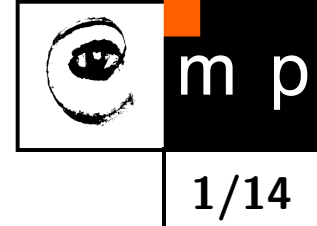


Center for Machine Perception presents

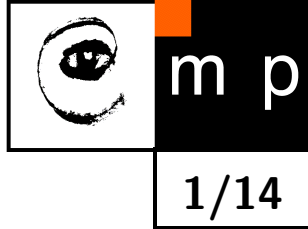


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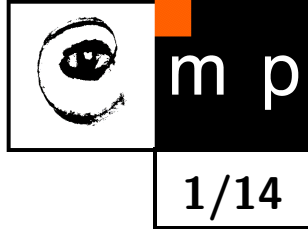
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**Semantical testing of tautologies in many-valued logics**

Center for Machine Perception presents



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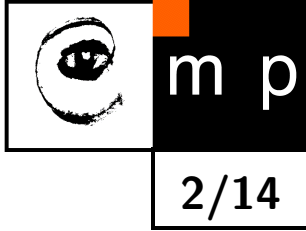
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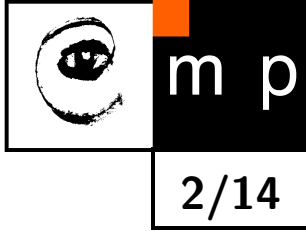
# Semantical testing of tautologies



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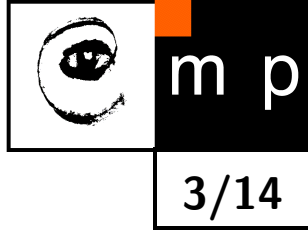
In **many-valued logics**:

Depends on the choice of many-valued logic;

the most interesting progress has been made in the Łukasiewicz logic, i.e., in MV-algebras

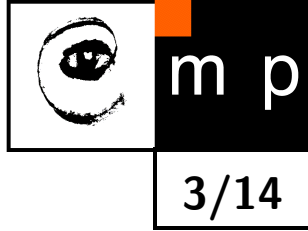
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$M$	number of truth values-1
1	16
2	65 536
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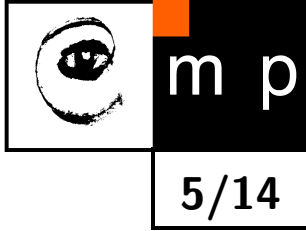
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$M \setminus n$	1	2	3
1	152		
2	2147 581 952	93 831 434 829 824	
3	$2.361 \cdot 10^{21}$	$1.081 \cdot 10^{32}$	$5.575 \cdot 10^{42}$

## 2nd bound



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Complexity:  $(b_1(M) + 1)^n$

$M \setminus n$	1	2	3	4	5
1	2				
2	3	9			
3	5	25	125		
4	9	81	729	6561	
5	17	289	4913	83 521	1419 857
6	33	1089	35 937	1185 921	39 135 393
7	65	4225	274 625	17 850 625	1160 290 625

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5	20	139	224	97	32
6	27	384	2024	2274	275
7	35	818	8280	25 332	12 200

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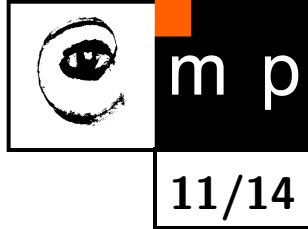
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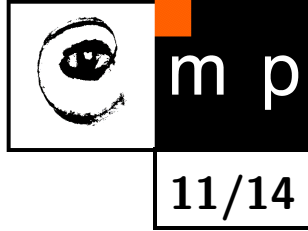
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# Semantical testing in many-valued logics 2



Related questions:

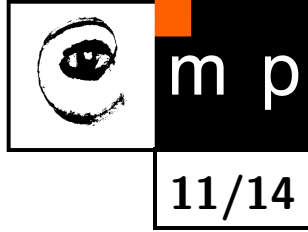
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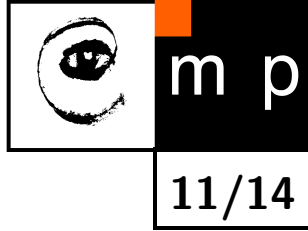
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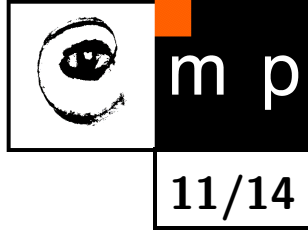


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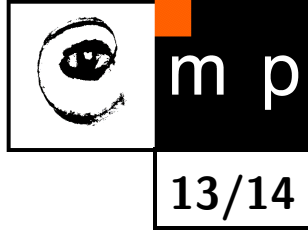
...

- Testing of tautologies in basic logic?

[Hájek; Haniková; Montagna, Pinna, and Tiezzi 03]; so far no implementation.

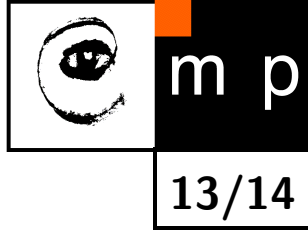
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# Semantical testing in many-valued logics 3



Alternative approaches to testing of tautologies:

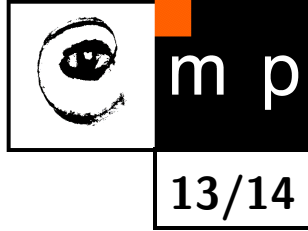
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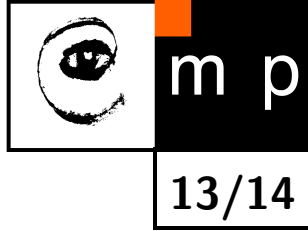


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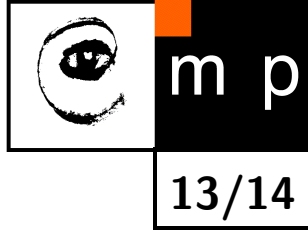
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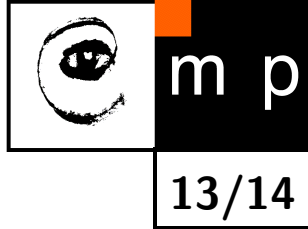
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Programmed by [Hähnle et al. ~95].

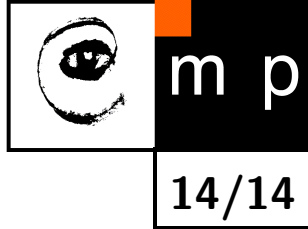
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# Semantical testing in many-valued logics 4



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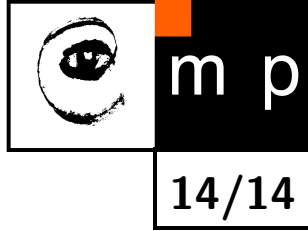
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The latter two methods do not guarantee an ultimate answer, but they give a reasonable chance to obtain it.