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Semantical testing of tautologies in many-valued logics

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What can computers do for us?

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What can computers do for us?

(And what they cannot do.)

Semantical testing of tautologies

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In **many-valued logics**:

Depends on the choice of many-valued logic;
the most interesting progress has been made in the Łukasiewicz logic, i.e., in MV-algebras

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  [Mundici 87] developed for another reason

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M ... the number of all occurrences of variables in the formula
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M	number of truth values-1
1	16
2	65 536
3	68 719 476 736
4	18 446 744 073 709 551 616
5	1267 650 600 228 229 401 496 703 205 376

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$M \backslash n$	1	2	3
1	152		
2	$2147\,581\,952$	$93\,831\,434\,829\,824$	
3	$2.361 \cdot 10^{21}$	$1.081 \cdot 10^{32}$	$5.575 \cdot 10^{42}$

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Complexity: $(b_1(M)+1)^n$

$M\backslash n$	1	2	3	4	5
1	2				
2	3	9			
3	5	25	125		
4	9	81	729	6561	
5	17	289	4913	83 521	1419 857
6	33	1089	35 937	1185 921	39 135 393
7	65	4225	274 625	17 850 625	1160 290 625

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2	5	4			
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4	14	54	35	16	
5	20	139	224	97	32
6	27	384	2024	2274	275
7	35	818	8280	25 332	12 200

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Implemented by [Brůžková 05].

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$\overset{\mathrm{S}}{\vee}$ increments M by 2 because $x \overset{\mathrm{S}}{\vee} y = (x \rightarrow y) \rightarrow y = \neg(\neg x \underset{\mathrm{S}}{\wedge} \neg y)$

Semantical testing in many-valued logics 2

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...
- Testing of tautologies in basic logic?
[Hájek; Haniková; Montagna, Pinna, and Tiezzi 03]; so far no implementation.

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Programmed by [Hähnle et al. ∼95].

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The latter two methods do not guarantee an ultimate answer, but they give a reasonable chance to obtain it.