What observables can be

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σ-fields [Kolmogorov]

σ-field (=**σ-algebra**) $\mathcal{L} \subseteq 2^{\Omega}$

- $\Omega \in \mathcal{L}$
- $A \in \mathcal{L} \implies \Omega \setminus A \in \mathcal{L}$
- $(A_n)_{n\in\mathbb{N}} \in \mathcal{L}^{\mathbb{N}} \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{L}$

State (=**probability measure**) $\mu\colon \mathcal{L} \to [0,1]$

- $\mu(\Omega) = 1$
- $(A_n)_{n\in\mathbb{N}} \in \mathcal{L}^{\mathbb{N}}$ mutually disjoint $\implies$
  $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$ (**σ-additivity**)

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Random variables vs. observables

[Kolmogorov] Random variable $f\colon \Omega \to \mathbb{R}$

- $\forall B \in \mathcal{B}(\mathbb{R}) : f^{-1}(B) = \{\omega \in \Omega : f(\omega) \in \mathcal{B}(\mathbb{R})\} \in \mathcal{L}$ (**$\mathcal{L}$-measurability**)

where $\mathcal{B}(\mathbb{R})$:= Borel σ-field on $\mathbb{R}$

[von Neumann 55, Varadarajan 68] Observable $x\colon \mathcal{B}(\mathbb{R}) \to \mathcal{L}$

- $x(\mathbb{R}) = \Omega$
- $(A_n)_{n\in\mathbb{N}} \in \mathcal{B}(\mathbb{R})^{\mathbb{N}} \implies x\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \bigcup_{n\in\mathbb{N}} x(A_n)$ (**σ-homomorphism**)

$$\begin{aligned} \forall B \in \mathcal{B}(\mathbb{R}) : \quad & x(B) &=& \; f^{-1}(B) \\ \forall \omega \in \Omega : \quad & f(\omega) &=& \; \text{unique } r \in \mathbb{R} : \omega \in x(\{r\}) \end{aligned}$$

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Expectation

For a random variable

$$\mathrm{E}_{\mathrm{RV}} f := \int_\Omega f \,\mathrm{d}\mu = \int_{\mathbb{R}} r \,\mathrm{d}F_f(r)$$

where $F_f(r) := \mu(f^{-1}((-\infty, r])) = P[f \leq r]$ is the cumulative distribution function of f

For an observable

$m_x(B) := \mu(x(B))$ defines a state $m_x := \mu \circ x$ on $\mathcal{B}(\mathbb{R})$ with the cumulative distribution function (of identity)

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Generalization 1: Boolean σ-algebras

Distributive σ-(ortho-)complemented bounded lattice $\mathcal{L}$

$$\emptyset := \mathbf{0} \qquad \Omega := \mathbf{1} \qquad \bigcup := \bigvee \qquad \bigcap := \bigwedge$$
$$\Omega \setminus A := A'$$

$' \colon \mathcal{L} \to \mathcal{L}$ is a (unique) lattice complementation, i.e.

$$A \wedge A' = \mathbf{0} \qquad A \vee A' = \mathbf{1}$$

Need not be isomorphic to σ-fields, but:

Loomis–Sikorski Theorem

Every Boolean σ-algebra is a σ-homomorphic image of a σ-field.

$\Longrightarrow$ the theory can be built in the Kolmogorovian style up to the Loomis–Sikorski homomorphism

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Example of a Boolean σ-algebra not isomorphic to a σ-field

$\mathcal{B}(\mathbb{R})$ factorized over all sets of Lebesgue measure 0.

Random variables cannot be defined,
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Generalization 2: σ-classes [Gudder 69]

Definition of a state requires closedness under **disjoint** unions

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States = σ-additive normalized measures

Observables = σ-homomorphisms

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Range of observable x

$$x(\mathcal{B}(\mathbb{R})) = \{x(B) : B \in \mathcal{B}(\mathbb{R})\}$$

is a sub-σ-field of $\mathcal{L}$

Observable x can be represented by an ($\mathcal{L}$-measurable) random variable $f\colon \Omega \to \mathbb{R}$

$\implies$ everything works as usual until we deal with only 1 observable, in particular:

- $\mathrm{E}_{\mathrm{OBS}}x = \int_{\mathbb{R}} r \,\mathrm{d}F_x(r)$
- $x \geq 0 \implies \mathrm{E}_{\mathrm{OBS}}x \geq 0$
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Noncompatibility

Problem

2 observables x, y; ranges $x(\mathcal{B}(\mathbb{R})), y(\mathcal{B}(\mathbb{R}))$ are sub-σ-fields of $\mathcal{L}$ which need not be contained in a common sub-σ-field of $\mathcal{L}$

$x(\mathcal{B}(\mathbb{R})), y(\mathcal{B}(\mathbb{R}))$ generate a σ-field $\mathcal{A} \subseteq 2^\Omega$, but it is possible that

- $B, C \in \mathcal{B}(\mathbb{R})$
- $x(B) \cup y(C) \in \mathcal{A} \setminus \mathcal{L}$

thus $\mathcal{A} \not\subseteq \mathcal{L}$

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Monotonicity

Problem [Gudder 76]

$$x \leq y \overset{?}{\Longrightarrow} \mathrm{E}_{\mathrm{OBS}}\, x \leq \mathrm{E}_{\mathrm{OBS}}\, y$$

Standard argument

$$\begin{aligned} z := y - x &\geq 0 \\ \mathrm{E}_{\mathrm{OBS}}\, z = \mathrm{E}_{\mathrm{OBS}}\, y - \mathrm{E}_{\mathrm{OBS}}\, x &\geq 0 \end{aligned}$$

This does not work because additivity is not guaranteed and observable $y - x$ need not exist!

Theorem [MN 84, Šipoš 85, D.T. Hoa 12]

Monotonicity holds.

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$$x + y = z \stackrel{?}{\Longrightarrow} \mathrm{E}_{\mathrm{OBS}}\, x + \mathrm{E}_{\mathrm{OBS}}\, y = \mathrm{E}_{\mathrm{OBS}}\, z$$

We must assume **summability**, i.e., the existence of observable $z = x + y$ and its expectation!

Theorem [Dravecký & Šipoš 80]

Additivity need not hold (for random variables with ranges $\mathbb{Z}$).

Theorem [Zerbe & Gudder 85, MN 89]

Additivity holds for observables with **finite** ranges.
(Weaker assumptions suffice, especially for 0-1 states [MN & P. Pták 83, MN 84].)

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Extensibility of states [Gudder 76, MN & P. Pták 83, Ovchinnikov 99, Sultanbekov 92 & 07]

Ranges of observables $x, y, z := x + y$ generate a σ-field $\mathcal{A}_{x,y,z}$, possibly $\not\subseteq \mathcal{L}$.

Theorem (extensibility $\implies$ additivity)

If $x, y, z := x + y$ are observables and state μ on $\mathcal{L}$ can be extended to $\mathcal{A}_{x,y,z}$, then additivity holds for x, y, z.

Proof

Everything reduces to the σ-field $\mathcal{A}_{x,y,z}$.

Problem [MN & P. Pták 83]

Extension of μ to a state on $\mathcal{A}_{x,y,z}$ need not exist, even if additivity holds.

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Extensibility and additivity

It suffices to extend μ to a **signed measure** on $\mathcal{A}_{x,y,z}$ (i.e., an $\mathbb{R}$-valued σ-additive function) [Gudder 76], provided that the respective expectations exist (correction by [MN & P. Pták 83]).

Theorem [Ovchinnikov 99]

For observables $x, y, z := x + y$ with **finite** ranges, μ can be extended to a **signed measure** on $\mathcal{A}_{x,y,z}$.

$\Longrightarrow$ Zerbe–Gudder Additivity Theorem

However, the proof uses the Zerbe–Gudder Additivity Theorem.

Problem

Find a direct proof of the above extensibility theorem, without additivity.

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σ-orthomodular poset (=σ-OMP)

bounded poset $\mathcal{L}$ with **orthocomplementation** $': \mathcal{L} \to \mathcal{L}$

- $A \leq B \implies B' \leq A'$
- $A'' = A$
- $A \wedge A' = \mathbf{0}$
- $A \leq B \implies B = A \vee (A' \wedge B)$ (**orthomodular law**)
- $(A_n)_{n\in\mathbb{N}} \in \mathcal{L}^{\mathbb{N}}$ mutually orthogonal $\implies \bigvee_{n\in\mathbb{N}} A_n \in \mathcal{L}$

where **mutually orthogonal** means $n \neq k \implies A_n \leq A'_k$
($A_n \wedge A_k = \mathbf{0}$ is not enough!)

- All suprema in the definition of a σ-OMP exist (can be obtained by orthogonal suprema and orthocomplements).
- A' is a lattice-theoretical complement of A, but not unique!
- Non-distributive, not a lattice

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σ-classes as σ-OMPs

$$\emptyset := \mathbf{0} \qquad \Omega := \mathbf{1} \qquad \bigcup := \bigvee \qquad \bigcap := \bigwedge$$
$$\Omega \setminus A := A'$$

Theorem [Gudder 76]

σ-OMP $\mathcal{L}$ is isomorphic to a σ-class
$\iff$ $\mathcal{L}$ has a **rich** space of 0-1 states
$\iff$ $(A \not\subseteq B \implies \exists$ state $\mu\colon \mathcal{L} \to \{0,1\} : \mu(A) = 1,\ \mu(B) = 0)$

Without set-representation,
random variables cannot be defined,
observables can.

Problem

Generalization of classical theorems about expectations (monotonicity, additivity, ...)

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Uniqueness problem

Question [Gudder 66]

Can 2 distinct observables have the same expectations in all states?

Answer 1 (trivial)

Yes, if the state space is small.
(It can be even empty [Greechie 71]; cf. [MN 94, H. Weber 94].)

Answer 2 (less trivial) [Gudder 66]

No, for observables with finite ranges and a rich state space.

Answer 3 (even less trivial) [Gudder 81]

Yes, for unbounded observables and a rich state space.

Answer 4 (highly nontrivial) [MN 95]

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Can 2 distinct observables have the same expectations in all states?

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$\mathcal{L}$:= the set of all closed subspaces of H

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States = σ-additive normalized measures

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Are there other states than mixtures of vector states?

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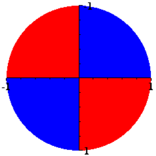
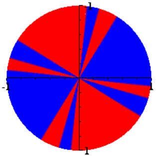
Yes, for $\dim H = 2$.





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No, for $\dim H \neq 2$.

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No, for 0-1 states and $\dim H \neq 2$.
Cf. [Lugovaja & Sherstnev 80, MN 04 & 09; Kochen & Specker 67, Svozil & Tkadlec, Mermin 93, Cabello 94, Peres 95, ...]

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For $\dim H = 3$, there is a **countable** sublattice admitting no 0-1 state.

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For dim $H \geq 3$, does the lattice of subspaces of the **rational** vector space admit a 0-1 state?

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Is there a nontrivial **$\mathbb{Z}_2$-valued measure** for dim $H \geq 3$?

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In σ-orthomodular posets, the **partial** operation of orthogonal join is applicable only to **distinct** orthogonal elements:

$$A \oplus B := \begin{cases} A \vee B & \text{if } A \leq B' \\ \text{undefined} & \text{otherwise} \end{cases}$$

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- $A \oplus B = B \oplus A$ (if one side is defined)
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Example: Hilbert effects

$\mathcal{L} := \{A : A \text{ self-adjoint operator on } H,\ \mathbf{0} \leq A \leq \text{id}\}$
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In particular, for $\dim H = 1$:

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Not exactly a special case of effect algebras because $\oplus$ is extended to a **total** operation.
Many classical theorems successfully generalized, e.g. laws of large numbers etc. using

Assumption: Independence [Riečan et al.]

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Problems of calculus of observables

Problem: Nondistributivity $\Longrightarrow$

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(A \odot B) \oplus (A \odot B') \oplus (A' \odot B) \oplus (A' \odot B') &\leq \mathbf{1} \\
\mu(A \odot B) + \mu(A \odot B') + \mu(A' \odot B) + \mu(A' \odot B') &\leq \mu(\mathbf{1})
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{1} = (\mu(A) + \mu(A')) \cdot (\mu(B) + \mu(B')) \\
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Thanks

<u>Thank you for your attention (if any).</u>