A General Solver Based on Sparse Resultants

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Motivation

- Problems from computer vision lead to polynomial system solving.
- Efficiency is needed for real time applications or for RANSAC framework.
- Current SOTA are Gröbner basis solvers.
  - Well known and fine-tuned for many years.
- Maybe for some problems different approach may be faster or more stable.

4*Problem	Size and stability								
	Our				GB				
	Solver size	Stability			Solver size		Stability		
	GEP	Mean	Median	Fail.(%)	GJ	Eigen	Mean	Median	Fail.(%)
Rel. pose 8pt + rad. dist.	29×29	-11.10	-11.89	0	32×48	16×16	-9.93	-10.45	0.86
Rolling shutter pose	20×20	-12.22	-12.38	0	47×55	8×8	-12.51	-12.70	0
Rel. pose 6pt + 1 sided rad. dist.	30×30	-11.12	-11.47	0	34×60	26×26	-11.20	-11.51	0
Rel. pose 6pt + 1 sided rad. dist. elim.	49×49	-9.38	-9.56	0	100×126	26×26	-9.64	-9.87	0
Abs. pose quiver	43×43	-12.00	-12.48	0	233×253	20×20	-10.12	-10.48	0.08
Generalized P4P + scale	18×18	-12.20	-12.34	0	47×55	8×8	-12.47	-12.67	0
Rel. pose 9pt + 2 rad. dist.	68×68	-10.29	-10.89	0	165×189	24×24	-8.92	-9.48	1.38
P4P + rad. dist.	28×28	-12.31	-12.53	0	140×156	16×16	-11.47	-11.74	0
Rel. pose 6pt + const. focal	18×18	-11.51	-11.95	0	31×50	15×15	-11.42	-11.90	0
Rel. pose 7pt. + 1 sided rad. dist. elim	35×35	-12.31	-12.53	0.02	51×70	19×19	-9.82	-10.14	0.16

Table: Comparison of some common minimal problems. Courtesy: Z. Kúkelová et al. Unpublished work in review.

Resultant of two univariate polynomials

Given $f, g \in \mathbb{C}[x]$

$$\begin{aligned} f &= a_0 x^l + \cdots + a_l, \qquad a_0 \neq 0 \qquad l > 0 \\ g &= b_0 x^m + \cdots + b_m, \qquad b_0 \neq 0 \qquad m > 0 \end{aligned} \tag{1}$$

the resultant $\mathrm{Res}(f, g)$ is

$$\mathrm{Res}(f,g) = \det \begin{bmatrix} & & & & a_0 & a_1 & a_2 & \cdots & a_l \\ & & & a_0 & a_1 & a_2 & \cdots & a_l & \\ & & a_0 & a_1 & a_2 & \cdots & a_l & & \\ & \iddots & \vdots & \vdots & \vdots & \iddots & & & \\ a_0 & a_1 & a_2 & \cdots & a_l & & & & \\ & & & & b_0 & b_1 & b_2 & \cdots & b_m \\ & & & b_0 & b_1 & b_2 & \cdots & b_m & \\ & & b_0 & b_1 & b_2 & \cdots & b_m & & \\ & \iddots & \vdots & \vdots & \vdots & \iddots & & & \\ b_0 & b_1 & b_2 & \cdots & b_m & & & & \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \\ a \end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \\ a \end{matrix}}\right\} l \text{ rows} \end{matrix} \tag{2}$$

The resultant vanishes iff f and g have a common root.

Example

$$\mathrm{Res}(x^3 + 4x - 1, 2x^2 + 3x + 7) = \det \begin{bmatrix} 0 & 1 & 0 & 4 & -1 \\ 1 & 0 & 4 & -1 & 0 \\ 0 & 0 & 2 & 3 & 7 \\ 0 & 2 & 3 & 7 & 0 \\ 2 & 3 & 7 & 0 & 0 \end{bmatrix} = 159 \quad (3)$$

No common root.

Multivariate example

Polynomials $f, g \in \mathbb{C}[x, y]$:

$$f = xy - 1, \tag{4}$$
$$g = x^2 + y^2 - 4. \tag{5}$$

Resultant with respect to x:

$$\operatorname{Res}(f, g) = \det \begin{bmatrix} 0 & y & -1 \\ y & -1 & 0 \\ 1 & 0 & y^2 - 4 \end{bmatrix} \tag{6}$$
$$= -y^4 + 4y^2 - 1. \tag{7}$$

Solve $-y^4 + 4y^2 - 1 = 0$ to find y-coordinates of the roots.

Resultant as a Macaulay matrix

$$\begin{aligned} f &= a_0 x^l + \cdots + a_l \qquad a_0 \neq 0 \qquad l > 0 \\ g &= b_0 x^m + \cdots + b_m \qquad b_0 \neq 0 \qquad m > 0 \end{aligned} \tag{8}$$

The Macaulay matrix M_d of degree $d = l + m$ is a coefficient matrix.

$$\begin{bmatrix} f \\ xf \\ x^2 f \\ \vdots \\ x^{d-m} f \\ g \\ xg \\ x^2 g \\ \vdots \\ x^{d-l} g \end{bmatrix} = M_d \begin{bmatrix} x^d \\ \vdots \\ x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{9}$$

$$\mathrm{Res}(f, g) = \det(M_d) \tag{10}$$

Sparse resultants

- Exploit sparsity of the given polynomials.
- Lead to resultants with smaller degrees.

For polynomials $f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_{n-1}]$:

$$\operatorname{Res}(f_1, \dots, f_n) = \det(M) \tag{11}$$

$$\begin{bmatrix} \vdots \\ \boldsymbol{x}^p f_{i_p} \\ \vdots \end{bmatrix} = M \begin{bmatrix} \vdots \\ \boldsymbol{x}^q \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \; q, p \in \mathcal{E}, i_p \in \{1, \dots, n\} \tag{12}$$

Two algorithms [1, 2] by I. Z. Emiris and J. F. Canny to obtain the matrix M.

[1] J. F. Canny, I. Z. Emiris. A Subdivision-Based Algorithm for the Sparse Resultant.
[2] I. Z. Emiris, J. F. Canny. Efficient Incremental Algorithms for the Sparse Resultant and the Mixed Volume.

Example on sparse polynomials

$$f = x^2y - x^2 + xy - x = 0 \tag{13}$$

$$g = x^2y + xy^2 + 2xy + 2y^2 = 0 \tag{14}$$

Bézout's bound: Number of solutions is at most equal to the product of the degrees.

- We expect at most 9 solutions.

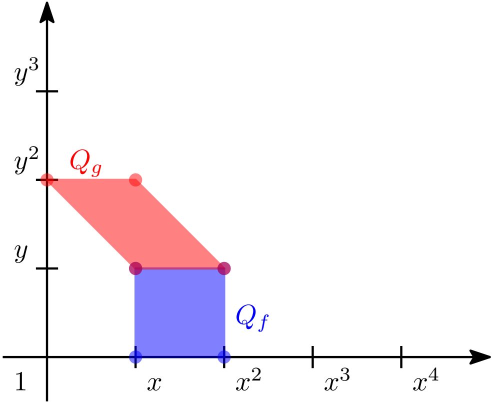
BKK bound: Sparse version of the Bézout's bound.

Example on sparse polynomials

$$f = x^2y - x^2 + xy - x = 0 \tag{13}$$

$$g = x^2y + xy^2 + 2xy + 2y^2 = 0 \tag{14}$$

Newton polytope Q_f: Convex hull of exponent vectors of monomials with nonzero coefficients in a polynomial f.



Example on sparse polynomials

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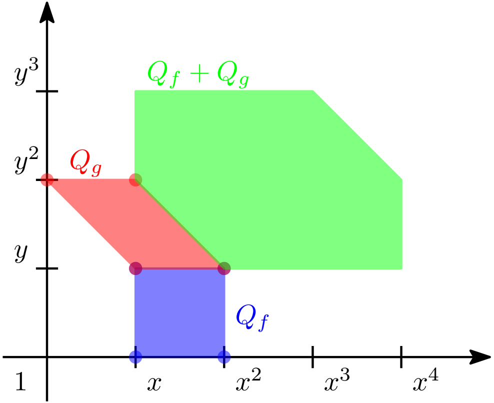
Minkowski sum:

$A + B = \{a + b \mid a \in A,\ b \in B\}$



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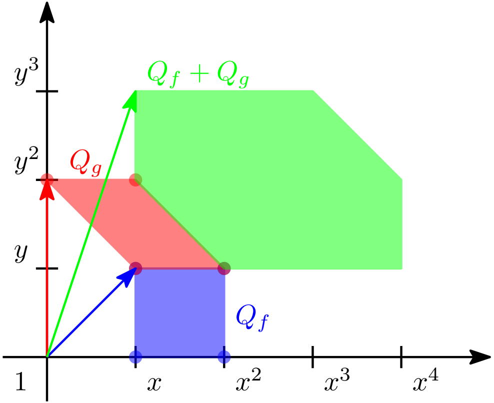

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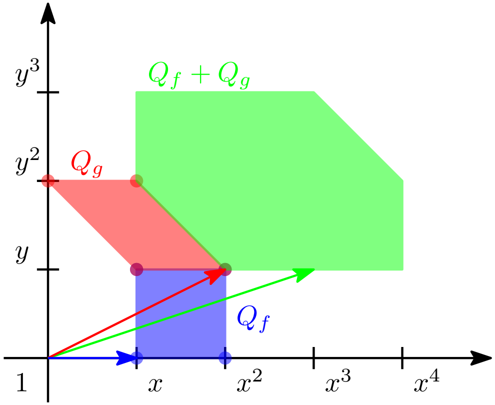


Example on sparse polynomials

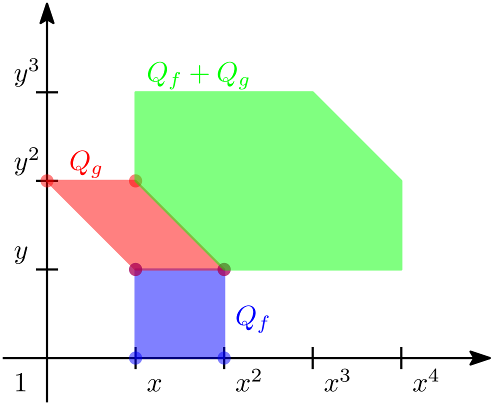
$$f = x^2y - x^2 + xy - x = 0 \quad (13)$$
$$g = x^2y + xy^2 + 2xy + 2y^2 = 0 \quad (14)$$

Mixed volume: In two-dimensional space leads to:

$$\mathrm{MV}(A, B) = \mathrm{Vol}(A + B) - \mathrm{Vol}(A) - \mathrm{Vol}(B).$$

In our example

$$\mathrm{MV}(Q_f, Q_g) = 5 - 1 - 1 = 3.$$



Example on sparse polynomials

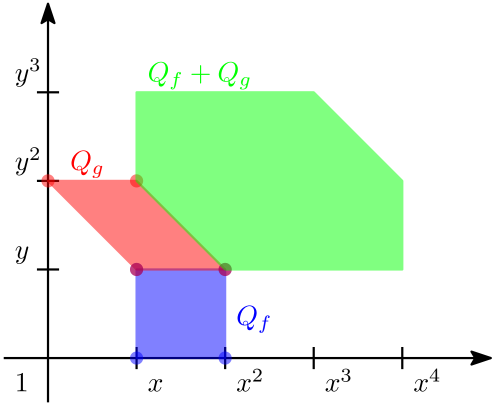
$$f = x^2y - x^2 + xy - x = 0 \quad (13)$$

$$g = x^2y + xy^2 + 2xy + 2y^2 = 0 \quad (14)$$

BKK bound: The number of nontrivial solutions is at most equal to $\mathrm{MV}(Q_{f_1}, \ldots, Q_{f_n})$.

In our example $\mathrm{MV}(Q_f, Q_g) = 3$, i.e. at most three nontrivial solutions.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$



Hiding a variable

- System of n polynomials in n variables, but forming a resultant requires an overconstrained system!
- Hide one of the variables (x_n) in the coefficient field. New variables are $\boldsymbol{x} = \begin{bmatrix} x_1 & \cdots & x_{n-1} \end{bmatrix}^\top$.

$$\begin{bmatrix} \vdots \\ \boldsymbol{x}^p f_{i_p}(\boldsymbol{x}, x_n) \\ \vdots \end{bmatrix} = M(x_n) \begin{bmatrix} \vdots \\ \boldsymbol{x}^q \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{15}$$

- Let $(\boldsymbol{\alpha}, \alpha_n)$ be a solution, then $M(\alpha_n)$ is singular and vector $\begin{bmatrix} \cdots & \boldsymbol{\alpha}^q & \cdots \end{bmatrix}^\top$ lies in the right kernel of $M(\alpha_n)$.

$$M(x_n) \tag{16}$$

$$\downarrow \quad \text{row and column permutation}$$

$$\begin{bmatrix} M_{11} & M_{12}(x_n) \\ M_{21}(x_n) & M_{22}(x_n) \end{bmatrix} \tag{17}$$

$$\downarrow \quad \text{Gaussian elimination}$$

$$\begin{bmatrix} M_{11} & M_{12}(x_n) \\ 0 & M'(x_n) \end{bmatrix} \tag{18}$$

Where $M'(x_n) = M_{22}(x_n) - M_{21}(x_n)M_{11}^{-1}M_{12}(x_n)$ is the Schur complement.

$$\begin{bmatrix} M_{11} & M_{12}(\alpha_n) \\ 0 & M'(\alpha_n) \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{v}' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

- $M'(\alpha_n)\boldsymbol{v}' = 0$ is a polynomial eigenvalue problem. Find $\boldsymbol{v}'$ and α_n.
- Then $\boldsymbol{v} = -M_{11}^{-1} M_{12}(\alpha_n)\boldsymbol{v}'$.
- Recover $\boldsymbol{\alpha}$ from $\begin{bmatrix} \boldsymbol{v} & \boldsymbol{v}' \end{bmatrix}^\top$.

Relative camera pose

- Minimal solver from 5 point matches.
- BKK bound on number of solution is 20.
- Eigendecomposition of matrix 20×20.
- Before D. Nistér [3] in 2004.

system	operation	CPU time
6×6	mixed volume	1m 16s
6×5	sparse resultant (offline)	12s
6×6 (first)	root finding (online)	0.2s
6×6 (second)	root finding (online)	1s (SUN SPARC 20)

Table: Camera motion from point matches: running times are measured on a DEC ALPHA 3000 except for the second system which is solved on a SUN SPARC 20.

[3] D. Nistér. An efficient solution to the five-point relative pose problem.

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