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Tomáš Pajdla
Tomáš Werner
Václav Hlaváč

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Czech Technical University, Faculty of Electrical Engineering
Department of Control Engineering, Center for Machine Perception
121 35 Prague 2, Karlovo náměstí 13, Czech Republic
fax +420 2 24357385, phone +420 2 24357465, http://cmp.felk.cvut.cz

ORIENTED PROJECTIVE RECONSTRUCTION[1)]

Tomáš Werner, Tomáš Pajdla, Václav Hlaváč[2)]

Abstract:
We introduce the notion of oriented projective reconstruction (OPR). We show that, in contrary to common belief, it is possible to obtain more than a projective reconstruction (PR) of a scene from uncalibrated real cameras, namely OPR. This is enabled by knowing that a real camera sees only points in front of it. The defining property of OPR is that the plane that is in an underlying Euclidean reconstruction at infinity does not intersect the convex hull of the reconstructed points in OPR. This is generally not true for PR. Thus, OPR can be viewed as a step between affine reconstruction (when the plane at infinity projects to infinity) and PR (when the position of the plane at infinity is unconstrained). The important practical consequence is that OPR preserves the convex hull, and the reconstructed scene is "topologically correct" and it can be e.g. rendered with hidden surfaces removed correctly.

1 Introduction

Let us consider reconstructing a scene from image points obtained by multiple cameras. If the cameras are calibrated, Euclidean reconstruction (ER) can be done [1]. If the calibration is unknown, only projective reconstruction (PR) can be done [4]. PR differs from ER in two basic things: (i) no Euclidean metrics is available, (ii) topology is different.

We introduce the notion of *oriented projective reconstruction* (OPR). The defining property of OPR is the existence of its orientation with respect to the plane at infinity. In more detail, the convex hull of a PR of a scene can generally be intersected by the plane that lies at infinity in the corresponding ER. This can cause the *order of some colinear sets of scene points to be changed* in the PR. In OPR, this plane is guaranteed to lie outside the convex hull of the scene points and hence the order is preserved.

We show that, in contrary to common belief, it is possible to obtain more a from uncalibrated

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[2)]Center for Machine Perception, Faculty of Electrical Engineering, Czech Technical University, Karlovo nám. 13, 12135 Prague, `http://cmp.felk.cvut.cz`

real cameras, namely OPR. It is enabled by the fact that the points observed by the cameras do not yield merely equations but also inequalities describing the knowledge that real cameras see only points *in front of them.* Using only equations for computing the scene structure yields PR, including also inequalities enables OPR.

OPR can be viewed as an intermediate step in the hierarchy of reconstructions [1], located between an affine reconstruction (when the the plane at infinity maps to the plane in infinity) and PR (when the plane at infinity maps to a general position). OPR has no metrics like PR, and topology of OPR is that of affine (or, equivalently, Euclidean) reconstruction.

Topological correctness of OPR (i.e., topology is equal to that of the underlying Euclidean reconstruction) has significant practical consequences. For instance, it enables rendering the OPR and removing hidden surfaces correctly, using a computer graphics algorithms for rendering a Euclidean model. Thiscannot be done with PR because scene points on a ray cannot be ordered by the distance from the camera center: if the plane at infinity intersects the scene, this order can be changed. This is not a well-known observation because e.g. Zhang writes [5]: *"Image-based rendering does have advantage over CAD-like modeling—usage of uncalibrated images. Then, only projective model can be obtained, and it is therefore difficult to use conventional rendering pipeline."*

In fact, when reconstructing from uncalibrated cameras we should always compute OPR rather than only PR. Necessary additional processing is neglectable and correct topology is obtained almost for free.

In the sequel, n-dimensional Euclidean space will be denoted by E_n and projective space by P_n. Scene points will be denoted by $\mathbf{x} \in E_3$ or $\mathbf{X} \in P_3$, image points in camera image plane by $\mathbf{u} \in E_2$, matrices of projective 3D-to-3D transformation by $\mathbf{H} \in R^{4\times4}$, and camera projection matrices by $\mathbf{M} \in R^{3\times4}$. We will always assume $\operatorname{rank}\mathbf{H} = 4$ and $\operatorname{rank}\mathbf{M} = 3$.

2 Definition of OPR

Let us have a scene $\mathcal{S}$ composed of I points. Points $\overline{\mathbf{x}}_i \in E_3, i = 1, \dots, I$, are called *Euclidean reconstruction (ER) of* $\mathcal{S}$ if they differ from the actual scene points by an unknown Euclidean transformation. Transforming all points $\overline{\mathbf{x}}_i$ by a common Euclidean transformation again yields ER of $\mathcal{S}$. We say that $\mathcal{S}$ is known up to a Euclidean transformation.

Points $\mathbf{X}_i \in P_3$ are called a *projective reconstruction* (PR) of $\mathcal{S}$ if

$$w_i \begin{pmatrix} \overline{\mathbf{x}}_i \\ 1 \end{pmatrix} = \mathbf{H}\mathbf{X}_i \tag{1}$$

where[1)] $w_i \neq 0$. Transforming $\mathbf{X}_i$ by a projective transformation yields again a PR of $\mathcal{S}$. We say that $\mathcal{S}$ is known up to a projective transformation $\mathbf{H}$.

Definition 1 (Oriented Projective Reconstruction) *Let us have a ER $\overline{\mathbf{x}}_i \in E_3$ of a scene $\mathcal{S}$. Let us have points $\hat{\mathbf{x}}_i \in E_3$, matrices $\overline{\mathbf{H}} \in R^{4\times 4}$, and numbers $\overline{w}_i$ and $\hat{\delta}$ so that for all $i = 1, \ldots, I$ it is*

$$\overline{w}_i \begin{pmatrix} \overline{\mathbf{x}}_i \\ 1 \end{pmatrix} = \overline{\mathbf{H}} \begin{pmatrix} \hat{\mathbf{x}}_i \\ 1 \end{pmatrix} \tag{2}$$

$$\hat{\delta}\overline{w}_i > 0 \tag{3}$$

Then points $\hat{\mathbf{x}}_i$ are called oriented projective reconstruction (OPR) of scene $\mathcal{S}$.

3 Relation between OPR and images from real cameras

3.1 Mathematical and real pinhole camera

Definition 2 (Mathematical and real pinhole camera) *Let us have a ER $\overline{\mathbf{x}}_i \in E_3$ of a scene $\mathcal{S}$ and a pinhole camera with a projection matrix $\overline{\mathbf{M}}$ which projects scene points $\overline{\mathbf{x}}_i$ to image points $\mathbf{u}_i \in E_2$ according to the following relation*[2)]*:*

$$\overline{z}_i \begin{pmatrix} \mathbf{u}_i \\ 1 \end{pmatrix} = \overline{\mathbf{M}} \begin{pmatrix} \overline{\mathbf{x}}_i \\ 1 \end{pmatrix} \tag{4}$$

The camera is called mathematical resp. real if $\overline{z}_i \neq 0$ resp. $\overline{z}_i > 0$ for all $i = 1, \ldots, I$.

This definition expresses the fact that a mathematical camera images points in the whole scene space, while a real camera images only points in front of its image plane.

[1)] We write points of ER as $\begin{pmatrix} \overline{\mathbf{x}}_i \\ 1 \end{pmatrix}$ without any loss of generality because we assume that $\mathcal{S}$ is a real scene in which all points are finite.

[2)] Image points in the camera's image plane are denoted by $\begin{pmatrix} \mathbf{u}_i \\ 1 \end{pmatrix}$ without any loss of generality because we assume that $\mathbf{u}_i$ are actually measured image points and therefore finite.

3.2 PR from correspondences in mathematical cameras

Let us have K mathematical cameras. A scene $\mathcal{S}$ is projected by these cameras to image points $\mathbf{u}_i^k, i = 1, \ldots, I, k = 1, \ldots, K$, where the point $\mathbf{u}_i^k$ is an image of i-th scene point in k-th camera. It is known [3, 2] that it is possible to find a PR of $\mathcal{S}$ from $\mathbf{u}_i^k$, i.e. to compute $\mathbf{X}_i$, $\mathbf{M}^k$, and $z_i^k \neq 0$ such that

$$z_i^k \begin{pmatrix} \mathbf{u}_i^k \\ 1 \end{pmatrix} = \mathbf{M}^k \mathbf{X}_i \tag{5}$$

It is also known that $\mathbf{M}^k$ and $\mathbf{X}_i$ are not determined uniquely—there exist different cameras $\mathbf{M}'^k$ and a different PR $\mathbf{X}'_i$ of $\mathcal{S}$ such that the images $\mathbf{u}_i^k$ remain the same, i.e.

$$z_i'^k \begin{pmatrix} \mathbf{u}_i^k \\ 1 \end{pmatrix} = \mathbf{M}'^k \mathbf{X}'_i \tag{6}$$

Points $\mathbf{X}_i$ and $\mathbf{X}'_i$ are related via a projective transformation $\mathbf{H}$ as

$$w_i \mathbf{X}_i = \mathbf{H} \mathbf{X}'_i \tag{7}$$

where $w_i \neq 0$. It means that it is possible to reconstruct $\mathcal{S}$ from the image points only up to an unknown projective transformation. The following theorem relates $z_i^k, z_i'^k, \mathbf{M}^k$ and $\mathbf{M}'^k$.

Theorem 1 *Let us have images $\mathbf{u}_i^k$ of points of a scene $\mathcal{S}$ and let us have two PR $\mathbf{X}_i, \mathbf{X}'_i$ of $\mathcal{S}$ for which relations (5), (6) and (7) hold. Then there exist numbers δ'^k such that for all $i = 1, \ldots, I$ and $k = 1, \ldots, K$*

$$\delta'^k \mathbf{M}'^k = \mathbf{M}^k \mathbf{H} \tag{8}$$

$$\delta'^k z_i'^k = w_i z_i^k \tag{9}$$

Proof. Substituting (7) to (5) yields

$$w_i z_i{}^k \begin{pmatrix} \mathbf{u}_i^k \\ 1 \end{pmatrix} = \mathbf{M}^k \mathbf{H} \mathbf{X}'_i \tag{10}$$

If this is to hold along with (6) for all $i = 1, \ldots, I$, it must be a multiple of (6). This multiple is generally different for each k, let us denote it by δ'^k. Comparing right-hand sides of (10) and (6) yields (8), comparing left-hand sides yields (9). □

3.3 OPR from correspondences in real cameras

Theorem 2 (Relation between OPR and images from real cameras) *Let us have a ER* $\overline{\mathbf{x}}_i$ *of a scene* $\mathcal{S}$ *and its projections* $\mathbf{u}_i^k$ *in* K *real cameras. Then for any* $\widehat{\mathbf{x}}_i, \widehat{\mathbf{M}}^k, \mathbf{u}_i^k$ *and* $\widehat{z}_i^k$ *such that*

$$\widehat{z}_i^k \begin{pmatrix} \mathbf{u}_i^k \\ 1 \end{pmatrix} = \widehat{\mathbf{M}}^k \begin{pmatrix} \widehat{\mathbf{x}}_i \\ 1 \end{pmatrix} \tag{11}$$

$$\widehat{z}_i^k > 0 \tag{12}$$

points $\widehat{\mathbf{x}}_i$ *are an OPR of* $\mathcal{S}$*.*

Proof. We want to prove that assumptions (11) and (12) of the theorem imply (2) and (3).

In section 3.2 we stated that points $\widehat{\mathbf{x}}_i$, for which (11) and $\widehat{z}_i^k \neq 0$ holds, are a PR of $\mathcal{S}$. Therefore they differ from ER of $\mathcal{S}$ by a projective transformation, and (11) implies (2).

According to Definition 2, there exist cameras $\overline{\mathbf{M}}^k$ and numbers $\overline{z}_i^k > 0$ for $\overline{\mathbf{x}}_i$ and $\mathbf{u}_i^k$ so that

$$\overline{z}_i^k \begin{pmatrix} \mathbf{u}_i^k \\ 1 \end{pmatrix} = \overline{\mathbf{M}}^k \begin{pmatrix} \overline{\mathbf{x}}_i \\ 1 \end{pmatrix} \tag{13}$$

According to (2), points $\overline{\mathbf{x}}_i$ and $\widehat{\mathbf{x}}_i$ are related via a projective transformation $\widehat{\mathbf{H}}$. Applying theorem 1 to relations (11), (13) and (2) yields that there are numbers $\widehat{\delta}^k$ such that

$$\widehat{\delta}^k \widehat{z}_i^k = \overline{w}_i \overline{z}_i^k \tag{14}$$

Since $\overline{z}_i^k > 0$ and $\widehat{z}_i^k > 0$, it is $\widehat{\delta}^k \overline{w}_i > 0$. It means that the sign of $\widehat{\delta}^k$ is independent on k and therefore there exist $\widehat{\delta}$ such that (3) holds. □

4 Construction of OPR

We will show how to construct an OPR from an existing PR. The following theorem is crucial for that.

Theorem 3 (Relation between OPR and PR) *Let us have a PR* $\mathbf{X}_i$ *of a scene* $\mathcal{S}$*, cameras* $\mathbf{M}^k$*, image points* $\mathbf{u}_i^k$*, and numbers* $z_i^k \neq 0$ *satisfying (5). Let us assume that a projective transformation* $\widehat{\mathbf{H}}$ *and numbers* δ^k *exist and satisfy the following condition:*

$$\delta^k z_i^k (\mathbf{n}_\infty^\top \widehat{\mathbf{H}} \mathbf{X}_i) > 0 \tag{15}$$

where $\mathbf{n}_\infty = (0,0,0,1)^\top$ *is a vector representing the plane at infinity* π_∞*. The points* $\widehat{\mathbf{x}}_i$ *given by the relation*

$$\widehat{w}_i \begin{pmatrix} \widehat{\mathbf{x}}_i \\ 1 \end{pmatrix} = \widehat{\mathbf{H}}\mathbf{X}_i \tag{16}$$

where $\widehat{w}_i > 0$*, are an OPR of* $\mathcal{S}$*.*

Proof. We are to prove that the assumptions of the theorem imply (2) and (3).

It is apparent from (16) that $\begin{pmatrix} \widehat{\mathbf{x}}_i \\ 1 \end{pmatrix} \in P_3$ is a PR of $\mathcal{S}$, hence (2) holds.

Scene points $\mathbf{X}_i$ are projected to image points $\mathbf{u}_i^k$ according to (5). Let points $\widehat{\mathbf{x}}_i$ be projected to the same image points by (11). It suffices to prove that $\widehat{z}_i^k > 0$, because then points $\widehat{\mathbf{x}}_i$ are an OPR of $\mathcal{S}$ according to Theorem 2.

Multiplying (16) by $\mathbf{n}_\infty^\top$ from the left yields $\widehat{w}_i \left[\mathbf{n}_\infty^\top \begin{pmatrix} \widehat{\mathbf{x}}_i \\ 1 \end{pmatrix}\right] = \mathbf{n}_\infty^\top \widehat{\mathbf{H}}\mathbf{X}_i$. In this expression, it holds $\mathbf{n}_\infty^\top \begin{pmatrix} \widehat{\mathbf{x}}_i \\ 1 \end{pmatrix} > 0$. Since (15) holds at the same time, it follows that

$$\delta^k z_i^k \widehat{w}_i > 0 \tag{17}$$

Applying theorem 1 to relations (11), (5) and (16) yields

$$\delta^k z_i^k = \widehat{w}_i \widehat{z}_i^k \tag{18}$$

By comparing (17) and (18) we obtain $\widehat{z}_i^k > 0$. □

The algorithm for constructing OPR from an existing PR, making use of this theorem, looks as follows:

1. Measure image points $\mathbf{u}_i^k \in E_2$ by real cameras, for $i = 1, \ldots, I$ and $k = 1, \ldots, K$.
2. Compute a PR $\mathbf{X}_i$ from $\mathbf{u}_i^k$, using existing algorithms [4].
3. Find a regular matrix $\widehat{\mathbf{H}}$ and numbers $\delta^k \neq 0$ satisfying the system of inequalities (15).
4. Compute OPR $\widehat{\mathbf{x}}_i$ by transforming PR $\mathbf{X}_i$ according to (16). Matrices $\widehat{\mathbf{M}}^k$ can be also computed using (8), if required.

Let us focus on how to solve step 3 in more detail. In (15), we know $\mathbf{X}_i$ and z_i^k from the PR. There remains unknowns $\widehat{\mathbf{H}}$ and δ^k. Only the signs of δ^k matter, their absolute value plays no role. We will show that once the sign of δ^k is determined for a single k, the remaining ones are uniquely determined by the signs of z_i^k. Let the sign of δ^1 be known. It is clear from (15) that

$$\delta^1 z_i^1 (\mathbf{n}_\infty^\top \widehat{\mathbf{H}} \mathbf{X}_i) \quad > \quad 0 \tag{19}$$

Therefore, finding $\widehat{\mathbf{H}}$ and δ^k satisfying (15) can be divided into two steps: (i) find some δ^1 and $\widehat{\mathbf{H}}$ satisfying (19), (ii) find $\delta^2, \ldots, \delta^k$ such that (15) holds for the found δ^1 and $\widehat{\mathbf{H}}$.

In the fact, condition (19) is concerned only with the fourth row $\widehat{\mathbf{h}}_4^\top$ of matrix $\widehat{\mathbf{H}}$ because $\mathbf{n}_\infty^\top \widehat{\mathbf{H}} \mathbf{X}_i = \widehat{\mathbf{h}}_4^\top \mathbf{X}_i$. The remaining rows can be chosen arbitrarily. Step (i) requires to find an intersection of halfspaces in the space of components of $\widehat{\mathbf{h}}_4$. The boundaries of these halfspaces are given by equations $\widehat{\mathbf{h}}_4^\top \mathbf{X}_i = 0$ and the orientations of these halfspaces are given by the signs of $\delta^1 z_i^k$. The sign of δ^1 can be chosen so that a solution for $\widehat{\mathbf{h}}_4$ exists.

The halfspaces have dimension four (the number of components of $\widehat{\mathbf{h}}_4$). However, if no point $\mathbf{X}_i$ is in the form $(0, 0, 0, \rho)^\top$ (which is true almost always or it can be forced by e.g. an appropriate transformation of $\mathbf{X}_i$), we can choose $\widehat{\mathbf{h}}_{44} = 1$ without any loss of generality and thus to reduce the dimension of the search space to three. The choice $\widehat{\mathbf{h}}_{44} = -1$ (i.e. multiplying $\widehat{\mathbf{H}}$ by -1) need not be taken into account because condition (19) remains satisfied after simultaneous changing the signs of $\widehat{\mathbf{H}}$ and δ^1, and therefore the choice $\widehat{\mathbf{h}}_{44} = 1$ is entirely general.

Thus, *finding all solutions δ^k and $\widehat{\mathbf{H}}$ of condition (15) requires searching for an intersection of halfspaces of dimension three.* However, *some* solution can be found in a simpler way, e.g. by a Monte Carlo method.

5 Properties of OPR

Let us denote

$$\mathcal{C}(\mathbf{x}_i) = \left\{ \mathbf{x} \in E_3 |\ \mathbf{x} = \sum_{i=1}^{I} \alpha_i \mathbf{x}_i,\ 0 \le \alpha_i \le 1,\ \sum_{i=1}^{I} \alpha_i = 1 \right\} \tag{20}$$

the convex hull of points $\mathbf{x}_1, \ldots, \mathbf{x}_I$. Let us denote the transformation, that assigns $\overline{\mathbf{x}}$ to $\widehat{\mathbf{x}}$ by solving (2) for $\overline{\mathbf{x}}$, as

$$h:\ \overline{\mathbf{x}} = h(\widehat{\mathbf{x}}) = \left[\overline{\mathbf{h}}_4^\top \begin{pmatrix} \widehat{\mathbf{x}} \\ 1 \end{pmatrix} \right]^{-1} \begin{pmatrix} \overline{\mathbf{h}}_1^\top \\ \overline{\mathbf{h}}_2^\top \\ \overline{\mathbf{h}}_3^\top \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{x}} \\ 1 \end{pmatrix} \tag{21}$$

where $\overline{\mathbf{h}}_i^\top$ are rows of $\overline{\mathbf{H}}$. The transformation h can be called *oriented projective transformation with respect to points* $\overline{\mathbf{x}}_i$. The OPR of $\mathcal{S}$ is characterized by the fact that it differs from ER of $\mathcal{S}$ by an oriented projective transformation.

The fundamental property of OPR is described in terms of the following property of h.

Theorem 4 (OPR preserves convex hull of ER) *Let us have ER* $\overline{\mathbf{x}}_i$ *and OPR* $\widehat{\mathbf{x}}_i$ *of a scene* $\mathcal{S}$*, related via the transformation* h *according to (21). Then it holds*

$$\mathcal{C}\left[h(\widehat{\mathbf{x}}_i)\right] = h\left[\mathcal{C}(\widehat{\mathbf{x}}_i)\right] \tag{22}$$

Proof. Equation (22) can be rewritten as

$$h\left(\sum_{i=1}^{I} \widehat{\alpha}_i \widehat{\mathbf{x}}_i\right) = \overline{\mathbf{x}} = \sum_{i=1}^{I} \overline{\alpha}_i h(\widehat{\mathbf{x}}_i) \tag{23}$$

We are to prove that the following two implications (corresponding to the equivalence (22)) hold for all I-tuples $\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_I$:

1. For all I-tuple $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_I$ such that $\sum_{i=1}^{I} \widehat{\alpha}_i = 1,\ 0 \leq \widehat{\alpha}_i \leq 1$ there exist an I-tuple $\overline{\alpha}_1, \ldots, \overline{\alpha}_I$ such that the conditions $\sum_{i=1}^{I} \overline{\alpha}_i = 1,\ 0 \leq \overline{\alpha}_i \leq 1$ and (23) hold.

2. For all I-tuple $\overline{\alpha}_1, \ldots, \overline{\alpha}_I$ such that $\sum_{i=1}^{I} \overline{\alpha}_i = 1,\ 0 \leq \overline{\alpha}_i \leq 1$ there exist an I-tuple $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_I$ such that the conditions $\sum_{i=1}^{I} \widehat{\alpha}_i = 1,\ 0 \leq \widehat{\alpha}_i \leq 1$ and (23) hold.

Let us prove only implication 1, implication 2 can be proved in a similar way. The conditions (23) and $\sum_{i=1}^{I} \overline{\alpha}_i = 1$ form a system of four linear equations for given $\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_I$ and $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_I$ which has I unknowns $\overline{\alpha}_1, \ldots, \overline{\alpha}_I$. We will show that this system has always a solution $\overline{\alpha}_1, \ldots, \overline{\alpha}_I$ such that $0 \leq \overline{\alpha}_i$.

Using the conditions $\sum_{i=1}^{I} \overline{\alpha}_i = 1,\ \sum_{i=1}^{I} \widehat{\alpha}_i = 1$, relation (23) can be rewritten to

$$\overline{w} \sum_{i=1}^{I} \overline{\alpha}_i \begin{pmatrix} \overline{\mathbf{x}}_i \\ 1 \end{pmatrix} = \overline{w} \begin{pmatrix} \overline{\mathbf{x}} \\ 1 \end{pmatrix} = \sum_{i=1}^{I} \widehat{\alpha}_i \overline{w}_i \begin{pmatrix} \overline{\mathbf{x}}_i \\ 1 \end{pmatrix} \tag{24}$$

Obviously, $\overline{\alpha}_i = \widehat{\alpha}_i \overline{w}_i / \overline{w}$ is a solution to this system. It is also a solution to the condition $\sum_{i=1}^{I} \overline{\alpha}_i = 1$ because the last row of the matrix equation (24) is $\overline{w} = \sum_{i=1}^{I} \widehat{\alpha}_i \overline{w}_i$. Since all $\overline{w}_i$ and $\overline{w}$ have equal signs (see (3)) and $0 \leq \widehat{\alpha}_i$, it must be also $0 \leq \overline{\alpha}_i$. Relations $\sum_{i=1}^{I} \overline{\alpha}_i = 1$ and $0 \leq \overline{\alpha}_i$ imply $\overline{\alpha}_i \leq 1$. □

This property of OPR shows that OPR is more than a mere PR. Table 1 shows a place of oriented projective transformation in the hierarchy of known transformations. Fig. 1

transformation	invariants
Euclidean	distance
affine	ratio of distances
oriented projective	cross ratio of distances, convex hull
projective	cross ratio of distances

Table 1: Hierarchy of transformations.

illustrates the hierarchy of reconstructions of a simple object. The situation is simplified by one dimension in this figure—the reconstructions are shown in 2-D space rather than in 3-D space.

It can be shown that the invariance of the convex hull implies *preserving the order of points at an arbitrary line segment inside the convex hull.* This property is important in practice, namely for rendering a model of a real scene, reconstructed from images captured by multiple cameras.

6 Using OPR: rendering an oriented projective model

If a model of a real scene is reconstructed from images captured by multiple calibrated cameras, a Euclidean model can be found. There are well-known algorithms for rendering a

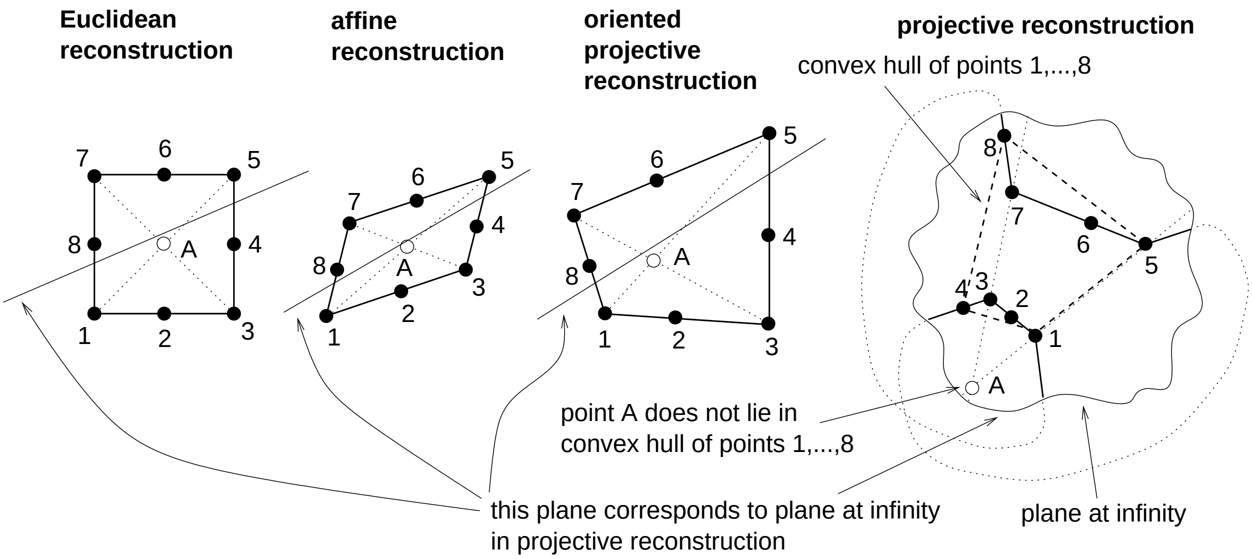

Figure 1: Hierarchy of reconstructions of a simple object. Projective reconstruction does not preserve convex hull of points $1, \ldots, 8$**.**

Euclidean model in computer graphics. There are graphics algorithms able to solve visibility e.g. by z-buffering, to warp texture, etc. Visibility is solved in the following way: the scene points at each camera ray are ordered according to the distance to the camera center, and only the closest point on each ray is rendered.

If the cameras are uncalibrated, ER (nor affine reconstruction) cannot be found without further knowledge about the cameras. It has been thought that at most a PR can be found [3, 2]. However, if a projective model is rendered by a rendering algorithm for a Euclidean model, a nonsense image will generally be obtained for the two following reasons:

1. *The image is be geometrically distorted.* This is caused by the unknown projective transformation between ER and PR.

2. *Visibility is solved incorrectly,* since the order of points at a ray in PR is generally not the same as in ER. Therefore, wrong scene points will be determined as the closest to the camera center.

The methods are known for computing the observer's projection matrix so that the geometrical distortion of the image is eliminated or limited to an acceptable level.

It has been thought [5] that a projective model cannot be rendered by a Euclidean rendering algorithm. *OPR allows it.* Since the order of points at rays is guaranteed to be the same as in the ER, visibility will be solved correctly.

References

[1] Olivier Faugeras. *Three-Dimensional Computer Vision.* MIT Press, 1993.

[2] Olivier D. Faugeras. What can be seen in three dimensions with an uncalibrated stereo rig? In *European Conference on Computer Vision*, pages 563–578, Santa Margherita Ligure, Italy, 1992. Springer Verlag.

[3] R. I. Hartley. Estimation of relative camera position for uncalibrated cameras. In *European Conference on Computer Vision*, pages 579–587, 1992.

[4] Charlie Rothwell, Gabriela Csurka, and Olivier Faugeras. A comparison of projective reconstruction methods for pairs of views. Technical Report 2538, INRIA, Sophia-Antipolis, France, April 1995. Available from `ftp.inria.fr`.

[5] Zhengyou Zhang. Image-based geometrically-correct photorealistic scene/object modelling (IBPhM): A review. In *Proc. Third Asian Conf. Comp. Vision, Hong Kong*, volume II, pages 340–349. Springer Verlag, January 8–10 1998.