

Constraint on Five Points in Two Images

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Abstract

It is well-known that epipolar geometry relating two uncalibrated images is determined by at least seven correspondences. If there are more than seven of them, their positions cannot be arbitrary if they are to be projections of any world points by any two cameras. Less than seven matches have been thought not to be constrained in any way. We show that there is a constraint even on five matches, i.e., that there exist forbidden configurations of five points in two images. The constraint is obtained by requiring orientation consistence—points on the wrong side of rays are not allowed. For allowed configurations, we show that epipoles must lie in domains with piecewise-conic boundaries, and how to compute them. We present a concise algorithm deciding whether a configuration is allowed or forbidden.

1. Introduction

The¹ retina of a central camera is usually modelled by the projective plane. This gives to our disposal synthetic projective geometry and also linear algebra because flats in the projective plane can be represented by subspaces of the real vector 3-space. However, it has been observed several times [5, 7, 13, 3] that projective geometry is an unnecessarily wide model for computer vision because it factorizes out orientation of flats. Orientation clearly plays a rôle because light rays *are* oriented halflines rather than unoriented infinite lines. Thus, the projective geometry can model configurations of flats and their camera projections that are inconsistent in orientation and hence cannot occur in reality.

The *oriented projective geometry* [9] (also called two-sided projective) is a more suitable model because it does represent orientation. In the vector space representation, orientation is expressed by orientation of bases of subspaces: two bases represent the same flat if they not only span the same subspace but also have equal handedness.

Epipolar geometry, so useful for photogrammetry and

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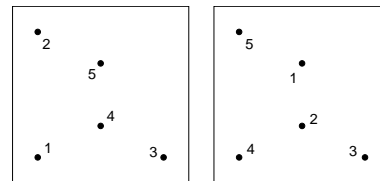


Figure 1. An example configuration of five points in two cameras for which no epipolar geometry exists.

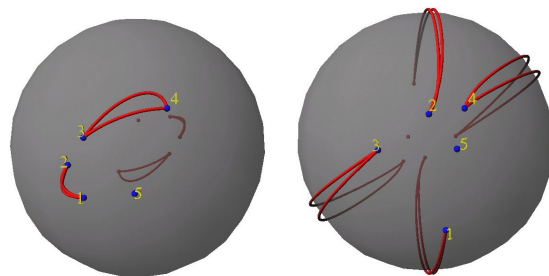


Figure 2. Two camera retinas with five corresponding points. Red curves delineate domains where the epipoles can only be located.

computer vision [6], has been studied already by the 19-th century projective geometers: given two 7-tuples of points in projective plane, find two other points such that the first point joined with the first 7-tuple and the second point joined with the second 7-tuple form a pair of line pencils related by a homography (e.g., in [4]). The well-known result is that 7 is the right number—for less points, the task is ambiguous, for more, it has no solution in general.

This paper shows that even five correspondences may have no solution for epipolar geometry if orientation is taken into account. Nothing is assumed about the cameras except the fact that they are central and linear—no affine information, no rectangular pixels, the cameras can be different (their image coordinate systems can even have opposite handedness), moreover they can be directional or panoramic². Figure 1 illustrates an example configura-

²Directional image is a set of points contained in a single hemisphere of the spherical 2-space. Panoramic (or omnidirectional) image does not have this property [13, 12]. Conventional cameras provide directional images.

tion of five correspondences that is not consistent with any epipolar geometry. Figure 2 shows an example of an allowed configuration and the loci where the epipoles can only be. These regions have piecewise-conic boundaries, with vertices in some of the given correspondences.

The paper is organized as follows. After introducing the notation, the oriented epipolar constraint and its decomposition into the projective and orientation part is described in section 3. Each of these parts imposes a constraint on simultaneous position of the epipoles, which combined together are promised to yield the final constraint. Section 4 derives the projective part and section 5 combines the two together. A concise explicit algorithm deciding whether a configuration of five point pairs is feasible is presented in section 6. Finally, classes of 3D reconstructions from 5 points are characterized in section 7.

2. Notation and Concepts

$\{x, y, z\}$ denotes a set while (x, y, z) denotes an ordered list. Set-theoretical difference is $A \setminus B$. \mathbb{R} is the set of reals, $\mathbf{x} \sim \mathbf{y}$ is equality of two vectors up to a non-zero scale, $\mathbf{x} \stackrel{\pm}{\sim} \mathbf{y}$ is equality up to a positive scale. $[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ is $\det([\mathbf{x}, \mathbf{y}, \mathbf{z}])$. Kronecker product of two vectors or matrices is $\mathbf{x} \otimes \mathbf{y}$, column-wise matrix vectorization is $\text{vec}(\mathbf{A})$.

\mathbb{P}^2 denotes the projective plane, and \mathbb{S}^2 the oriented projective 2-space (i.e., the 2-sphere). Following [9], $\neg x$ denotes the antipode x , $x \vee y$ denotes the join of two points forming a line, and $a \wedge b$ is the meet of two lines forming a point. $x \vee y$ is often abbreviated as xy . The pencil of three lines joining points x_1, x_2, x_3 with y is thus $y(x_1, x_2, x_3)$.

By x in italics we denote a point, either from \mathbb{P}^2 or \mathbb{S}^2 . A homogeneous vector from \mathbb{R}^3 representing x is denoted by \mathbf{x} in bold face. All vectors $\{\lambda \mathbf{x} \mid \lambda \neq 0\}$ represent the same $x \in \mathbb{P}^2$, while only vectors $\{\lambda \mathbf{x} \mid \lambda > 0\}$ represent $x \in \mathbb{S}^2$. The antipode $\neg x$ is represented by $-\mathbf{x}$. We will freely swap between boldface and italics symbols, assuming automatically that they relate to the same object. Note, symbols in italics mean abstract geometrical entities in the synthetic geometry sense, whereas boldface symbols are mere vectors. Thus expressions like $x \in y \vee z$ or $\mathbf{x} + \mathbf{y}$ are meaningful, whereas $x + y$ or $\mathbf{x} \in \mathbf{l}$ are not.

For any symbol s , s denotes entity related to the first image, while s' is related to the second image.

$(x_1, \dots, x_n) \bar{\wedge} (y_1, \dots, y_n)$ means that two point lists from \mathbb{P}^n are related by a homography, $\exists \mathbf{H} \forall i (\mathbf{y}_i \sim \mathbf{H} \mathbf{x}_i)$. $(x_1, \dots, x_n) \bar{\wedge}_+ (y_1, \dots, y_n)$ means that two point lists from \mathbb{S}^n are related by an oriented homography [9, 13], $\exists \mathbf{H} \forall i (\mathbf{y}_i \stackrel{\pm}{\sim} \mathbf{H} \mathbf{x}_i)$. In \mathbb{P}^1 , the cross-ratio of four points is $\langle x_1, x_2, x_3, x_4 \rangle = [\mathbf{x}_1, \mathbf{x}_2][\mathbf{x}_3, \mathbf{x}_4][\mathbf{x}_1, \mathbf{x}_3]^{-1}[\mathbf{x}_2, \mathbf{x}_4]^{-1}$. Symbol $x_1 x_2 \parallel x_3 x_4$ denotes that x_1, x_2 separates x_3, x_4 . There are 3 possibilities, $x_1 x_2 \parallel x_3 x_4$, $x_1 x_3 \parallel x_2 x_4$, and $x_1 x_4 \parallel x_2 x_3$, respectively equivalent to $\langle x_1, x_2, x_3, x_4 \rangle$ greater than 1, between 0 and 1, and negative [8].

Chirotope $\chi[(x_1, \dots, x_n)]$ of a list of points from \mathbb{S}^2 is the list of signs $\text{sgn}[\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k]$ where i, j, k are all different triplets of indices from $1, \dots, n$. *Chirotope* $\chi[y(x_1, \dots, x_n)]$ of the pencil of lines $y(x_1, \dots, x_n)$ is the list of signs $\text{sgn}[y, \mathbf{x}_i, \mathbf{x}_j]$ for all different index pairs i, j . *Chirotope* is a concept from the theory of oriented matroids [2] (also cf. [3]). It captures exactly the orientation information which is factorized out by the projective geometry.

3. Two Constraints on Joint Epipole

Let $G^n = \{(x_1, x'_1), \dots, (x_n, x'_n)\}$ be a configuration of n point pairs in two images. A necessary condition for G^n to be projections of some n world points by some pair of cameras is that [4, 6] there are two points e and e' such that

$$e(x_1, \dots, x_n) \bar{\wedge} e'(x'_1, \dots, x'_n). \quad (1)$$

This condition is not sufficient because it allows for scene points on wrong sides of the rays. It can be augmented to forbid that as follows: the points and epipoles are considered in \mathbb{S}^2 rather than in \mathbb{P}^2 , and *the homography must be oriented*³, $e(x_1, \dots, x_n) \bar{\wedge}_+ e'(x'_1, \dots, x'_n)$. We will not use the last relation directly; instead, it can be shown [13] equivalent to simultaneously requiring (1) and

$$\chi[e(x_1, \dots, x_n)] = -\chi[e'(x'_1, \dots, x'_n)]. \quad (2)$$

For $n \leq 7$, the unoriented condition (1) alone does not impose any constraint on either epipole separately. However, it is a constraint on the *joint epipole* $(e, e') \in \mathbb{P}^2 \times \mathbb{P}^2$ for $n \geq 2$. I.e., if e is given, the location of e' might not be arbitrary anymore, forced to obey (1). Similarly, (2) constrains the oriented joint epipole $(e, e') \in \mathbb{S}^2 \times \mathbb{S}^2$.

We will refer to (1) and (2) as *projective constraint* and *orientation constraint* on the joint epipole, respectively, and denote the sets of joint epipoles satisfying them for a configuration G^n respectively by $P(G^n)$ and $S(G^n)$. The set satisfying both constraints is⁴ $E(G^n) = P(G^n) \cap S(G^n)$.

The rest of the paper will show that for $n = 5$, the combined constraint restricts e to lie in a subset of the first image plane rather than anywhere in the image, i.e., $\{e \mid \exists e' [(e, e') \in E(G^5)]\} \neq \mathbb{P}^2$, and that there are configurations for which $E(G^5) = \emptyset$.

4. Projective Constraint

This section characterizes $P(G^n)$ for up to $n = 5$. All points are considered in \mathbb{P}^2 .

³The analytical form of this *oriented epipolar constraint* is $[e']_{\times} \mathbf{x}'_i \stackrel{\pm}{\sim} \mathbf{F} \mathbf{x}_i$ where $\mathbf{F} = [e]_{\times} \mathbf{H}$ is the fundamental matrix [11, 13, 12].

⁴For notation simplicity, we are slightly informal in the paper about whether P and S take and return entities from \mathbb{P}^2 or \mathbb{S}^2 . Naturally, P should take a list of point pairs from \mathbb{P}^2 and return a subset of $\mathbb{P}^2 \times \mathbb{P}^2$, and S should take a list of point pairs from \mathbb{S}^2 and return a subset of $\mathbb{S}^2 \times \mathbb{S}^2$. Depending on the context, we assume that P and S work with either space. It does make sense because we can either identify antipodes, or use the fact that (2) is invariant to replacing (e, e') with $(-e, -e')$.

Line-preserving property. In $P(G^2)$, both epipoles are arbitrary unless one of them lies on the line joining the points. It follows from (1) that $(e \in x_1 \vee x_2) \Leftrightarrow (e' \in x'_1 \vee x'_2)$. This indeed holds for all pairs of correspondences in G^n .

Four points. In $P(G^4)$, one epipole is free and the other epipole is constrained by it to lie on a conic. Let e be given. Let Q^e denote the conic determined by the points $\{e, x_1, \dots, x_4\}$ and h denote the homography sending (x_1, \dots, x_4) to (x'_1, \dots, x'_4) . Then e' must lie on the conic $Q'^e = h(Q^e)$. This follows from the Steiner theorem [8], saying that the locus of points e for which $\langle e(x_1, \dots, x_4) \rangle$ is constant is any conic passing through x_1, \dots, x_4 .

Five points. In $P(G^5)$, one epipole is free and the other is related with it by a *Cremona transformation of degree 5*. This is a classical result [10], which however is not widely known in computer vision⁵. It is obtained from the fact that the four-point constraint must hold for all five four-tuples from the five correspondences, i.e., $P(G^5) = \cap_{i=1}^5 P(G_i^4)$ where $G_i^4 = \{(x_1, x'_1), \dots, (x_5, x'_5)\} \setminus \{(x_i, x'_i)\}$.

Let e be given and let Q_i^e be the conic through the points $\{e, x_1, \dots, x_5\} \setminus \{x_i\}$ and h_i be the homography sending x_j to x'_j for $j \in \{1, \dots, 5\} \setminus \{i\}$. Then e' must lie simultaneously on all conics $Q_i'^e = h_i(Q_i^e)$. Quite surprisingly, all $Q_1'^e, \dots, Q_5'^e$ intersect in a single point⁶ for any e . Thus, there is a functional relationship between e and e' defined everywhere except in x_i , which is denoted by $e' = f(e)$.

Let us construct six conics Q_0, \dots, Q_5 in the first image and six conics Q'_0, \dots, Q'_5 in the second image as follows. Q_0 is given by x_1, \dots, x_5 and $Q'_0 = h_0(Q_0)$. The conics Q'_0 and Q_1, \dots, Q_5 are obtained symmetrically.

If $e \in Q_0$ (i.e., e is conconic with all x_i), then $Q_i^e = Q_0$ and $Q_i'^e = Q'_0$ for all i , and e' lies on the common intersection of Q'_i . We will denote this intersection by x'_0 . Symmetrically, x_0 is the common intersection of Q_i . The new pair (x_0, x'_0) is uniquely defined by G^5 and it is the only point pair which would bring no information if added to G^5 . In other words, $G_{\text{deg}}^6 = G^5 \cup \{(x_0, x'_0)\}$ is a degenerate configuration in the sense that $P(G^5) = P(G_{\text{deg}}^6)$. The new pair (x_0, x'_0) plays the same rôle as any other pair (x_i, x'_i) . In fact, the extended configuration G_{deg}^6 is more complete and symmetric than the original G^5 . The pair (x_0, x'_0) also

⁵I thank Thomas Buchanan [1] for telling me about [10].

⁶This is shown in [10] and follows also from analytic characterization of $P(G^n)$. Analytically, $P(G^n)$ can be obtained from n conditions on the fundamental matrix induced by the point pairs, $\mathbf{x}_i^\top \mathbf{F} \mathbf{x}_i = 0$, and one condition $\mathbf{F} \mathbf{e} = \mathbf{0}$. These can be combined to the single system $\mathbf{A} \text{vec}(\mathbf{F}) = \mathbf{0}$, where $\mathbf{A} = [\mathbf{x}_1 \otimes \mathbf{x}_1 \mid \dots \mid \mathbf{x}_n \otimes \mathbf{x}_n \mid \mathbf{e} \otimes \mathbf{I}_{3 \times 3}]^\top$. $P(G^n)$ is given by solvability of this system as follows. The allowed positions of e must satisfy the condition $\text{rank}(\mathbf{A}) < 9$. Given an allowed e , the allowed positions of e' are given by solving the system for \mathbf{F} and hence for e' . In particular, for $n = 5$, the size of \mathbf{A} is 8×9 , therefore $\text{rank}(\mathbf{A}) = 8$ and any position of e is allowed; and e yields a unique (up to scale and for non-degenerate G^5) solution for \mathbf{F} and hence for e' .

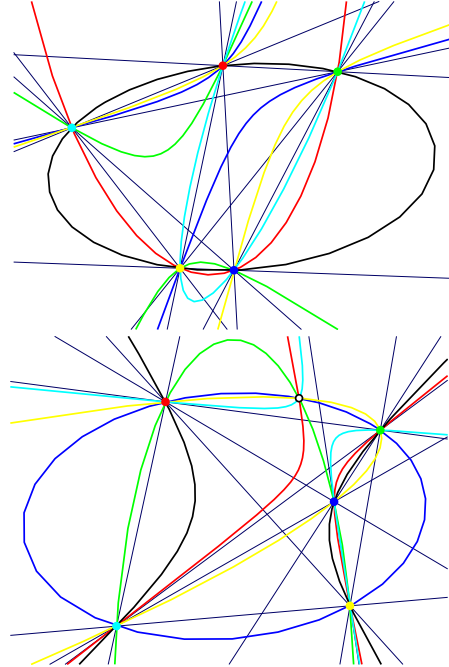


Figure 3. Exceptional points and curves of f . Each index is depicted in different color. Also, lines $x_i x'_j$ are depicted.

allows to construct Q_i and Q'_i without using h_i , because Q_i is a conic through $\{x_0, \dots, x_5\} \setminus \{x_i\}$, and similarly for Q'_i .

Recall [8] that Cremona transformations are mappings from \mathbb{P}^2 to \mathbb{P}^2 that are birational, i.e., both forward and inverse mapping is given by polynomials in homogeneous coordinates. A Cremona transformation of degree d is continuous and invertible everywhere except in $d + 1$ *exceptional points* where it is undefined, and $d + 1$ *exceptional curves*, which are each sent to the corresponding exceptional point. Everything is symmetric, there are $d + 1$ exceptional points and curves in each plane.

$f(e)$ can be evaluated by computing the intersection e' of any two conics $Q_i'^e$. Since these two conics share three points, the intersection can be expressed by polynomials in homogeneous coordinates⁷. Therefore f is birational. We showed that⁸ $f(Q_i) = x'_i$ for each $i = 0, \dots, 5$, therefore Q_i and Q'_i are its exceptional curves, and x_i and x'_i are its corresponding exceptional points.

⁷The intersection of two conics sharing three points is conveniently computed using the *reciprocal Cremona transformation* [8]. The point reciprocal to $\mathbf{x} = (x^1, x^2, x^3)^\top$ is given by $\mathbf{x}^* = (x^2 x^3, x^3 x^1, x^1 x^2)^\top$. Note, $(\mathbf{x}^*)^* = \mathbf{x}$. Its crucial property is that it sends the line $x_1 x_2$ to the conic given by $\{e_1, e_2, e_3, x_1^*, x_2^*\}$, where \mathbf{e}_i are the three basis vectors $(1, 0, 0)^\top, (0, 1, 0)^\top, (0, 0, 1)^\top$. The fourth common point of two conics $\{e_1, e_2, e_3, x_1, x_2\}$ and $\{e_1, e_2, e_3, y_1, y_2\}$ is just $(x_1^* x_2^* \wedge y_1^* y_2^*)^*$. By simplifying this expression I also obtained that the degree of f is 5.

⁸The following subtlety needs to be explained here. If $e \in Q_i$, then $e' = f(Q_i) = x'_i$. But this does not correspond to any configuration of cameras and a 3D point X_i corresponding to (x_i, x'_i) , because if $e' = x'_i$ then the camera centers and X_i are colinear and inevitably it is $e = x_i$, if it happens to be consistent with (1). The statement $f(Q_i) = x'_i$ describes the *limit case*, it says what is happening to e' if e is very near Q_i .

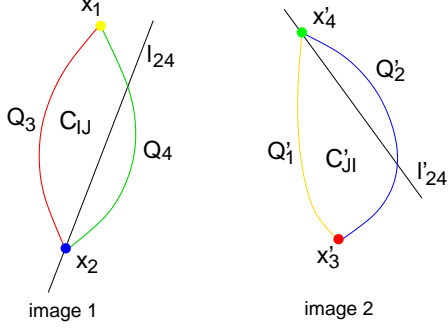


Figure 4. Two cells C_{IJ} and C'_{JI} , where $I = \{3, 4\}$ and $J = \{1, 2\}$. Q_3, Q_4 correspond respectively to x'_3, x'_4 , and x_1, x_2 correspond to Q'_1, Q'_2 . If there is a line l_{ij} crossing C_{IJ} , there is the corresponding line l'_{ij} crossing C'_{JI} . Function f maps C_{IJ} to C'_{JI} and Q_3, Q_4 to x'_3, x'_4 , respectively.

Cell-preserving property. Central to G^5 is the conic web in each image plane formed by the exceptional curves and points of f (figure 3). This structure has a number of interesting properties, most of them naturally following from the correspondence induced by f and from continuity of f . The most useful for us is the following cell-preserving property.

The exceptional curves partition the first image plane into a system of *cells*, delineated by piecewise-conic boundaries. A cell C is uniquely characterized by the subset of conics that participate to form its boundary, or, equivalently, by the subset of points x_i that belong to its boundary. Let $I, J \subset \{0, \dots, 5\}$ be two sets of indices and let C_{IJ} denote a cell whose boundary is formed by arcs of conics $\{Q_i | i \in I\}$ and by points $\{x_j | j \in J\}$.

For each cell C_{IJ} in the first image there is always a cell $C'_{I'J'}$ in the second image such that $I' = J$ and $J' = I$. Therefore, there is one-to-one correspondence between cells in the first and the second image. The cell-preserving property says that any cell C_{IJ} is sent to its corresponding cell C'_{JI} by f . A typical pair of two corresponding cells is illustrated in figure 4.

The properties of f we will need can be summarized as follows: $f(x_i x_j) = x'_i x'_j$, $f(Q_i) = x'_i$, and $f(C_{IJ}) = C'_{JI}$.

5. Combining the Two Constraints

In this section, the set $E(G^5) = P(G^5) \cap S(G^5)$ is constructed. The following observation is crucial: if two epipoles in the first image plane are inside a single cell C_{IJ} , they are either both allowed or both forbidden. Formally,

$$(e_1, e_2 \in C_{IJ}) \Rightarrow [(e_1, f(e_1)) \in E \Leftrightarrow (e_2, f(e_2)) \in E].$$

In other words, the property of ‘being an allowed epipole’ is constant inside any cell. This is rather surprising—one would expect that this property changes if the epipole e crosses any line $x_i x_j$, because the sign of $[e, x_i, x_j]$ in (2) swaps. However, due to the line preserving property of f ,

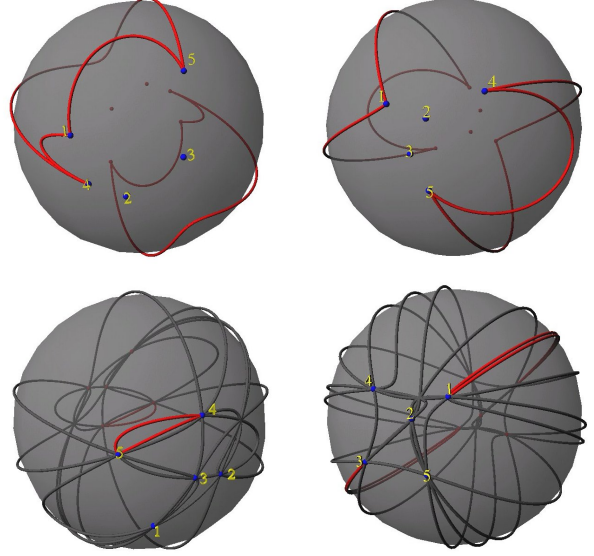


Figure 5. Two example image pairs with computed regions of allowed epipoles. Each row depicts one pair. The second example also shows the full family of six conics in grey.

the sign of $[e', x'_i, x'_j]$ swaps simultaneously. Thus, $E(G^5)$ consist of pairs of *allowed cells*.

If $e' = x'_i$, the projective constraint restricts e to lie on Q_i . Imposing the oriented constraint further forces e to lie on a segment of Q_i , rather than anywhere on Q_i . We call it the *allowed segment*. The allowed segment of Q_i is given by omitting the index i from (2), i.e., by the condition

$$(\forall j, k \in \{1, \dots, 5\} \setminus \{i\}) ([e, x_j, x_k][x'_i, x'_j, x'_k] < 0) \quad (3)$$

where e is any point on the segment. Four segments, delimited by neighboring pairs of points $\{x_1, \dots, x_5\} \setminus \{x_i\}$, are considered on each Q_i . It can be shown that there is *at most one* allowed segment on each Q_i .

It follows from continuity of f that each allowed segment is part of the boundary of some allowed cell and that the boundary of each allowed cell consists only of allowed segments. Two allowed cells never touch, except in point x_0 ; however, if an allowed segment contains x_0 it never stops there but always passes through it because x_0 is absent in (3). In other words, x_0 is the only point where allowed segments can cross each other. Therefore, the set of allowed segments uniquely determines the set of allowed cells and hence $E(G^5)$. Constructing $E(G^5)$ has just been reduced to enumeration of allowed segments, which is easy using (3).

Examples of regions of allowed epipoles for three different configurations are in figures 2 and 5. An example of a forbidden configuration is in figure 1, in which x_1, \dots, x_5 are respectively $(0, 0, 1)^\top$, $(0, 1, 1)^\top$, $(1, 0, 1)^\top$, $(\frac{1}{2}, \frac{1}{4}, 1)^\top$, $(\frac{1}{2}, \frac{3}{4}, 1)^\top$, and x'_1, \dots, x'_5 are respectively equal to x_4, x_5, x_3, x_2, x_1 .

6. Explicit Constraint on Five Points

Here, we'll give the algorithm deciding whether a given configuration G^5 is forbidden or allowed, i.e., whether $E(G^5)$ is empty or non-empty. It already follows from section 5, but we will simplify it, providing additional insights.

How to sort five points on a conic. We will show how to compute the order of five points on the conic given by them. We will need it later in this section.

The set of possible orderings of n points in \mathbb{P}^1 is equal to the set of their permutations factorized by cyclic permutations and reversion. This factorized set has $n!/(2n)$ elements, which we call *projective orders*. E.g., there is a single projective order of 3 points, $\{(123)\}$, 3 projective orders of 4 points, $\{(1234), (1243), (1423)\}$, and 12 projective orders of 5 points,

$$\begin{aligned} &\{(12345), (12354), (12534), (15234), \\ &\quad (12435), (12453), (12543), (15243), \\ &\quad (14235), (14253), (14523), (15423)\}. \end{aligned}$$

A d -degree curve in \mathbb{P}^d is, up to a d -dimensional projectivity, isomorphic to \mathbb{P}^1 via the mapping $c: \mathbb{P}^1 \rightarrow \mathbb{P}^d$ given by $(u, v)^\top \mapsto (u^d, u^{d-1}v, \dots, uv^{d-1}, v^d)^\top$. Examples are a conic in \mathbb{P}^2 or a (twisted) cubic in \mathbb{P}^3 . Therefore, the projective order of points $c(x_1), \dots, c(x_n) \in \mathbb{P}^d$ on the curve is the same as the projective order of $x_1, \dots, x_n \in \mathbb{P}^1$.

The projective order of $d+3$ points on a d -degree curve in \mathbb{P}^d can be computed without explicitly using c because it is closely related to separations of pencils of quadruples of hyperplanes. E.g., there exist 12 'qualitatively different' configurations of 5 points in \mathbb{P}^2 , given by whether $x_i(x_j, x_k) \parallel x_i(x_l, x_m)$ for all different 5-tuples (i, j, k, l, m) . They correspond to the 12 projective orders of the points on the conic given by them.

In \mathbb{P}^2 , it is easy to verify that given four points x_1, \dots, x_4 on a conic Q , the following three statements are equivalent:

1. The projective order of x_1, \dots, x_4 on Q is (1234).
2. Any point $e \neq x_i$ on Q satisfies $e(x_1, x_3) \parallel e(x_2, x_4)$.
3. The point $x_1x_3 \wedge x_2x_4$ belongs to the interior of Q .

The equivalence $1 \Leftrightarrow 2$ yields the projective order of points x_1, \dots, x_4 on the conic given by x_1, \dots, x_5 . The projective order of all points can be found by testing separations for several permutations of the points. It's a bit tricky, the following MATLAB function does it for arbitrary d :

```
function o = porderd(x)
o = 1:3;
for j = 4:size(x,2)
while 1
i = po4(x(:, [o(1:3) j o(4:end) j+1:end]));
if i < 3, break, end
o = o([2:end 1]);
end
o = [o(1:i) j o(i+1:end)];
```

end

```
function i = po4(x)
r=det(x(:, [1 2 5:end]))*det(x(:, [3 4 5:end]))/...
det(x(:, [1 3 5:end]))/det(x(:, [2 4 5:end]));
if r>1 i=1; elseif r<0 i=2; else i=3; end
```

In \mathbb{S}^2 , five points define a *spherical conic* rather than a projective one. A non-degenerate spherical conic Q consists of two disconnected antipodal components (see figure 6a). The equivalence $1 \Leftrightarrow 3$ allows to find out *which point is on which component* (up to swapping the components) as follows. The *inscribed rectangle theorem* says that the line $l = (x_1x_2 \wedge x_3x_4)(x_1x_4 \wedge x_2x_3)$ is the polar of the point $y = x_1x_3 \wedge x_2x_4$ with respect to any conic passing through x_1, \dots, x_4 . If the projective order of x_1, \dots, x_4 on Q is (1234), which can be done by renaming indices, y is inside Q and hence l is outside. Thus, l separates the two components of Q in \mathbb{S}^2 (figure 6b). The points on the right of l are on one component, the points on the left on the other.

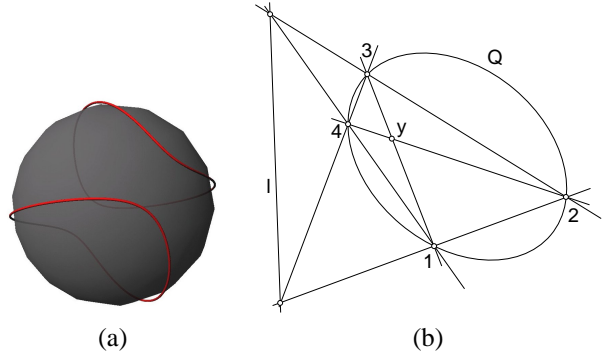


Figure 6. Conic in \mathbb{S}^2 consists of two disconnected components (a). If the projective order of points is (1234), point $13 \wedge 24$ is inside Q and line $(12 \wedge 34)(14 \wedge 23)$ outside (b).

Simplifying the algorithm. Recall from section 5 that finding whether Q_i contains an allowed segment requires computing an arbitrary point e on each of the four segments of Q_i and testing it for (3). We can get rid of computing these arbitrary points as follows.

Rename indices of the pairs $(x_1, x'_1), \dots, (x_5, x'_5)$ so that the projective order of x'_1, \dots, x'_5 on Q'_0 is (12345). Since $Q_i = h_i^{-1}(Q'_0)$ and projectivities preserve projective order, this ensures that the projective order of x_2, x_3, x_4, x_5 on Q_1 is (2345), of x_1, x_3, x_4, x_5 on Q_2 is (1345), etc.

Since $E(G^n)$ is invariant to replacing any pair (x_i, x'_i) with $(-x_i, -x'_i)$, we can do this replacement for appropriate pairs so that $\{x_1, \dots, x_5\} \setminus \{x_i\}$ all lie on a single component of the spherical conic Q_i , using the pivot line as described above. Now, the list of signs of $[e, \mathbf{x}_j, \mathbf{x}_k]$, where e is on any fixed segment and $j, k \in \{1, \dots, 5\} \setminus \{i\}$, is independent on the positions of x_1, \dots, x_5 and hence known in advance. Testing whether Q_i contains an allowed segment merely requires comparing these signs with signs of the appropriate determinants $[\mathbf{x}'_i, \mathbf{x}'_j, \mathbf{x}'_k]$ according to (3).

E.g., after renaming indices and replacing appropriate pairs with antipodes, the conic Q_5 with points x_1, \dots, x_4 on it looks always as shown in figure 6b. It is obvious from the figure what are the signs of $[e, \mathbf{x}_j, \mathbf{x}_k]$ for each segment.

To summarize, the following function returns whether a configuration of five point pairs in two images is allowed:

```
function OK = test_epi_5points(x1,x2)
o = porderd(x2); x1 = x1(:,o); x2 = x2(:,o);
for i = 1:5
    OK = testQ5(x1,x2);
    if OK, break, end
    p = [2:5 1]; x1 = x1(:,p); x2 = x2(:,p);
end

function OK = testQ5(x1,x2)
l1 = cross(x1(:, [1 3 1 3]), x1(:, [2 4 4 2]));
l1 = cross(l1(:, [1 3]), l1(:, [2 4]));
l1 = cross(l1(:, 1), l1(:, 2))';
x2 = x2 .* ([1;1;1]* (l1*x1));
s1 = [-1 -1 -1 +1 +1 +1
      +1 -1 -1 -1 -1 +1
      +1 +1 -1 +1 -1 -1
      +1 +1 +1 +1 +1 +1]';
s2 = sign([det(x2(:, [5 1 2]))
          det(x2(:, [5 1 3]))
          det(x2(:, [5 1 4]))
          det(x2(:, [5 2 3]))
          det(x2(:, [5 2 4]))
          det(x2(:, [5 3 4]))]) * [1 1 1 1]);
OK = any( all(s1==s2) | all(s1== -s2) );
```

Note, given all but one points from G^5 , e.g., $\{x_1, \dots, x_5, x'_1, \dots, x'_4\}$, the locus of the last point, x'_5 , for which $E(G^5) \neq \emptyset$ is a *non-convex spherical polygon*. Its vertices are a subset of the $5 + 15 = 20$ intersections of the $\binom{5}{2}$ lines $x'_i x'_j$ (not only x'_i) and its sides are segments of these lines.

7. Chirotope of 3D Reconstruction

If $(e, e') \in E(G^5)$, there exist an *oriented projective reconstruction* from G^5 , i.e., there are two cameras \mathbf{P}, \mathbf{P}' and 3D points X_i such that $\mathbf{x}_i \stackrel{\pm}{\sim} \mathbf{P}\mathbf{X}_i$ and $\mathbf{x}'_i \stackrel{\pm}{\sim} \mathbf{P}'\mathbf{X}_i$ [5, 6, 13]. Let \mathbf{P}^\wedge denote the *oriented camera center*, being the wedge product of the three rows of \mathbf{P} . In this section, we will investigate the 3D reconstruction $(\mathbf{P}^\wedge, \mathbf{P}'^\wedge, \mathbf{X}_1, \dots, \mathbf{X}_5)$.

Consider the extended degenerate configuration of six points $G_{\text{deg}}^6 = G^5 \cup \{(x_0, x'_0)\}$, which was described in section 4. Let all 12 points x_i, x'_i be from \mathbb{S}^2 and let (e, e') satisfy (1) and (2). Unlike in section 5, also the pair (x_0, x'_0) is assigned orientation and is present in (2). We can compute, up to a 3D oriented homography, cameras \mathbf{P} and \mathbf{P}' and points X_0, \dots, X_5 such that $\mathbf{x}_i \stackrel{\pm}{\sim} \mathbf{P}\mathbf{X}_i$ and $\mathbf{x}'_i \stackrel{\pm}{\sim} \mathbf{P}'\mathbf{X}_i$ for $i = 0, \dots, 5$. I noticed that if e crosses the line $x_i x_j$, two following determinants change sign: $[\mathbf{P}^\wedge, \mathbf{P}'^\wedge, \mathbf{X}_i, \mathbf{X}_j]$ and $[\mathbf{X}_a, \mathbf{X}_b, \mathbf{X}_c, \mathbf{X}_d]$ where $\{a, b, c, d\} = \{0, \dots, 5\} \setminus \{i, j\}$.

Thus, the lines joining the points x_0, \dots, x_5 partition the allowed cell(s) into *regions corresponding to different chirotopes* of the 3D reconstruction $(\mathbf{P}^\wedge, \mathbf{P}'^\wedge, \mathbf{X}_1, \dots, \mathbf{X}_5)$ from G^5 .

Theses classes of 3D reconstructions can be tested for *cheiral inequalities*⁹, because the solvability of cheiral inequalities clearly depends only on the chirotope of the reconstruction. This provides another constraint on G^5 , which, however, can be violated only for panoramic images [13], because it has been shown that the cheiral inequalities are always satisfied for two directional images [5, 13].

8. Conclusion

The new constraint on five points in two images has been presented, based on existing epipolar constraint and orientation consistence, in contrary to the common belief that any configuration of five point pairs corresponds to some (infinitely many) epipolar geometries. We have not discussed the form of the constraint for degenerate configurations G^5 .

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⁹Cheiral inequalities [5, 6, 13] are another constraint on G^n . It requires that either $\{\mathbf{P}^\wedge, \mathbf{P}'^\wedge, \mathbf{X}_1, \dots, \mathbf{X}_n\}$ or $\{-\mathbf{P}^\wedge, -\mathbf{P}'^\wedge, \mathbf{X}_1, \dots, \mathbf{X}_n\}$ fits to a single 3-hemisphere of \mathbb{S}^3 .